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# The existence of two closed geodesics on every Finsler 2-sphere

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**Abstract** In this paper, we prove that for every Finsler metric on $S^2$ there exist at least two distinct prime closed geodesics.

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## Contents



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## 1 Introduction

The study of closed geodesics on spheres is a classical and important problem both in dynamical systems and differential geometry. The results of Franks [12] in 1992 and Bangert [5] in 1993 prove that for every Riemannian metric on $S^2$ there exist infinitely many geometrically distinct closed geodesics. On the other hand, in 1973 A. Katok constructed his remarkable examples of irreversible Finsler metrics on $S^2$ that possess precisely two distinct prime closed geodesics (cf. [44] for further explanation), which are inverse curves of each other.

The main result of this paper is:

**Theorem 1.1** *For every Finsler metric on the 2-sphere, there exist at least two distinct prime closed geodesics.*

Note that in Theorem 1.1 the Finsler metric is not assumed to be reversible. In particular, if $c : S^1 = \mathbf{R}/\mathbf{Z} \to S^2$ is a closed geodesic then its inverse curve $c^{-1}$, defined as $c^{-1}(t) = c(-t)$, will not be a geodesic in general. If it is, it is counted as a second closed geodesic, as in Katok's examples. For the definitions of Finsler metrics and their geodesics we refer to [7] and [39].

For the case of the two-spheres, Theorem 1.1 solves a problem posed by Anosov in his address [1] to the ICM in 1974.

According to the classical theorem of Lyusternik-Fet [34] from 1951, there exists at least one closed geodesic on every compact Riemannian manifold. The proof of this theorem, see e.g. [4] or [24], is variational and carries over to the Finsler case. The authors are aware of only few results on the existence of more than one closed geodesic on Finsler 2-spheres. In [36] of 1989, Rademacher proved the existence of a second closed geodesic on a Finsler 2-sphere provided all its closed geodesics—including iterates—are non-degenerate. If, in addition, for every hyperbolic closed geodesic the stable and unstable manifolds intersect transversely, then the results by Hofer, Wysocki and Zehnder published in 2003 in [20] imply the existence of either two or infinitely many closed geodesics. This alternative also holds for Finsler metrics with $K \geq 1$ for which every geodesic loop is longer than $\pi$. This was proved in 2006 by Harris and Paternain [16]. Their proof is based on [19]. Recently, Rademacher [38] proved the existence of two closed geodesics on Finsler 2-spheres satisfying a pinching condition on the flag curvature. We refer readers also to [11] of Fet in 1965 for a result in the non-degenerate reversible Finsler case.

It is tempting to try and prove Theorem 1.1 along the lines of the proof of the celebrated theorem of Lyusternik-Schnirelmann on the existence of three closed geodesics without self-intersections on every Riemannian 2-sphere, cf. [2,13,21,32,33,41]. This would be possible if one could find an energy-decreasing deformation on the space of closed curves without self-intersections that works for irreversible Finsler metrics on $S^2$. As observed by Rademacher, see p. 82 of [36], Katok's examples show that such a deformation does not exist. Instead, our proof is by contradiction. We assume that there is only one closed geodesic $c$ and make a case by case study according to the different symplectic normal forms of the linearized Poincaré map of $c$. For more details see the outline of the proof in the next section.

Using Legendre transformation one can reformulate Theorem 1.1 as a result on convex Hamiltonian systems on the cotangent bundle $T^*S^2$ of $S^2$ as follows.

**Theorem 1.2** *Let $H : T^*S^2 \to \mathbf{R}$ be smooth and assume that the restrictions of $H$ to the fibers of the cotangent bundle $T^*S^2 \to S^2$ have positive definite Hessian everywhere and attain their minima. Let $r$ be a real number such that the sublevel $H^{-1}((-\infty, r)) \subseteq T^*S^2$ contains the zero section. Then the Hamiltonian system $X_H$ determined by $H$ and the standard symplectic structure on $T^*S^2$ has at least two periodic orbits on the level surface $H^{-1}(r)$.*

## 2 Outline of the proof

We first explain how we count closed geodesics on a Finsler manifold $(M, F)$. If $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is a closed geodesic, then so are its *iterates* $c^m : S^1 \to M$ defined by $c^m(s) = c(ms)$, for all positive integers $m$. The closed geodesic $c$ is called *prime* if there does not exist $\tilde{c} : S^1 \to M$ such that $c = \tilde{c}^m$ for some $m \geq 2$. Two prime closed geodesics $c$ and $d$ are *distinct* if they do not only differ by translation of parameter, i.e., if there does not exist $\theta \in \mathbf{R}/\mathbf{Z}$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}/\mathbf{Z}$. Now the number of closed geodesics of $(M, F)$ is defined as the (possibly infinite) number of distinct prime closed geodesics of $(M, F)$. In order to prove Theorem 1.1 we argue by contradiction and assume that there is a Finsler 2-sphere $(S^2, F)$ with only one prime closed geodesic $c$. The closed geodesics $\{c^m | m \geq 1\}$ are critical points of the Finsler energy functional $E : \Lambda \to \mathbf{R}$, $E(\gamma) = \frac{1}{2}\int_{S^1} F^2(\dot{\gamma}(t))dt$ on the space $\Lambda = \Lambda S^2$ of closed $H^1$-curves. As critical points of $E$ the closed geodesics have an index

$$i(c^m) = index(D^2E(c^m))$$

and a nullity

$$\nu(c^m) = nullity(D^2E(c^m)) - 1.$$

As proved in [29] (cf. [31]), there are nine possible cases for the sequences $\{i(c^m)\}_{m\geq 1}$ and $\{\nu(c^m)\}_{m\geq 1}$ depending on the different symplectic normal forms of the linearized Poincaré map of $c$. Note that here by [28] the iteration formulae in [29] work for Morse indices of closed geodesics on Riemannian and Finsler manifolds. In most of these nine cases we show that Morse Theory imposes the existence of an additional closed geodesic besides $c$. Here, the two non-degenerate cases, in which $\nu(c^m) \equiv 0$, have already been treated in [36]. In some of the degenerate cases the results by Hingston, [17] and [18], are of great use. In others, we can use arguments of Gromoll-Meyer type, cf. [14] and [15], to obtain contradictions to the Morse inequalities with $\mathbf{Q}$-coefficients.

Here we need very precise information on the homology created by iterated closed geodesics in the loop space. This is the content of Sect. 3. It relies on Rademacher's Habilitationsschrift [37], while part of it is new. In some of these cases the existence of a second closed geodesic also follows from the recent paper [38] by Rademacher. There remains one subcase, treated in Sect. 10, in which the preceding methods fail. To obtain a contradiction in this subcase, we use on the one hand a result by Rademacher from [37] on the mean index of $c$, cf. Theorem 5.2 below, and on the other hand a detailed study on how the first and second homologies of the free loop space modulo point curves should be created by the closed geodesic and its iterates, cf. Sect. 10.

We add some remarks concerning notations in this paper.

For $a \in \mathbf{R}$ we set $[a] = \max\{k \in \mathbf{Z} | k \leq a\}$. By $\mathbf{N}$ we denote the set of positive integers. Throughout the paper, homology modules will be with respect to coefficients in $\mathbf{Q}$. This allows us to use the transfer theorem for the homology of spaces with $\mathbf{Z}_m$-actions, see Lemma 3.6.

## 3 Critical modules of iterated closed geodesics

In this section, we review part of the Morse theory of the energy functional on the free loop space of a Finsler manifold. Some facts that we need do not yet exist in the literature, and so we present some of the details. On a compact Finsler manifold $(M, F)$ we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ with a natural structure of Riemannian Hilbert manifold on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries. Here a map $c : S^1 \to M$ is $H^1$ if it is absolutely continuous and its derivative $\dot{c}$ is square integrable, cf. [24, Chapters 1 and 2]. The $S^1$-action is defined by translating the parameter, i.e.

$$(s \cdot \gamma)(t) = \gamma(t + s) \tag{3.1}$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric $F$ defines an energy functional $E$ and a length functional $L$ on $\Lambda$ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 \mathrm{dt}, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) \mathrm{dt}. \tag{3.2}$$

Both functionals are invariant under the $S^1$-action. For $\kappa \in \mathbf{R}$ we set $\Lambda^\kappa = \{\gamma \in \Lambda \mid E(\gamma) \leq \kappa\}$ and $\Lambda^{\kappa-} = \{\gamma \in \Lambda \mid E(\gamma) < \kappa\}$.

The critical points of $E$ of positive energy are precisely the closed geodesics $c : S^1 \to M$ of the Finsler structure. If $F$ is not Riemannian, then due to the non-differentiability of $F^2$ on the zero section, the energy $E : \Lambda \to [0, \infty)$ is not smooth, but only of class $C^{1,1}$, cf. [35]. If $c \in \Lambda$ is a closed geodesic, then $c$ is a regular curve, i.e. $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the second differential $E''(c)$ of $E$ at $c$ exists.

As usual we define the index $i(c)$ of $c$ as the maximal dimension of subspaces of $T_c\Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of $c$ so that $\nu(c) + 1$

is the dimension of the null space of $E''(c)$. The relations between $E''(c)$, the index form and Jacobi fields are analogous to the Riemannian case, see e.g. [39].

For $m \in \mathbf{N}$ we denote the $m$-fold iteration map $\phi_m : \Lambda \to \Lambda$ by

$$\phi_m(\gamma)(t) = \gamma(mt), \quad \forall \gamma \in \Lambda, t \in S^1. \tag{3.3}$$

We also use the notation $\phi_m(\gamma) = \gamma^m$. Note that $\phi_m$ is an embedding satisfying

$$E \circ \phi_m = m^2 E \tag{3.4}$$

and

$$\phi_m(ms \cdot \gamma) = s \cdot \phi_m(\gamma), \quad \forall \gamma \in \Lambda, s \in S^1. \tag{3.5}$$

According to (3.4), if $c$ is a closed geodesic and $\xi, \eta \in T_c\Lambda$, then

$$E''(c^m)(D\phi_m(\xi), D\phi_m(\eta)) = m^2 E''(c)(\xi, \eta). \tag{3.6}$$

Since the null space of $E''(c)$ is the space of periodic Jacobi fields along $c$, one easily concludes:

**Lemma 3.1** *$D\phi_m(c)$ maps the null space of $E''(c)$ injectively into the null space of $E''(c^m)$.*

If $\gamma \in \Lambda$ is not constant, then the multiplicity $m(\gamma)$ of $\gamma$ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$, then $\gamma$ is called prime. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

For a closed geodesic $c$ we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}.$$

If $A \subseteq \Lambda$ is invariant under some subgroup $\Gamma$ of $S^1$, we denote by $A/\Gamma$ the quotient space of $A$ with respect to the action of $\Gamma$.

Using singular homology with rational coefficients we will consider the following critical $\mathbf{Q}$-modules of a closed geodesic $c \in \Lambda$:

$$\begin{cases} C_*(E, c) & = H_*\left(\Lambda(c) \cup \{c\}, \Lambda(c)\right), \\ C_*(E, S^1 \cdot c) & = H_*\left(\Lambda(c) \cup S^1 \cdot c, \Lambda(c)\right), \\ \overline{C}_*(E, c) & = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right). \end{cases} \tag{3.7}$$

In order to relate the critical modules to the index and nullity of $c$ we would like to use the results by Gromoll and Meyer from [14,15]. Unfortunately, this is not directly possible, since in general the functional $E$ will not be of class $C^2$ in any neighborhood of $c$. Following [37, Section 6.2], we will evade this problem by introducing finite-dimensional approximations to $\Lambda$. We choose an arbitrary energy value $a > 0$ and $k \in \mathbf{N}$ such that every $F$-geodesic of length $< \sqrt{2a/k}$ is minimal. Then

$$\Lambda(k, a) = \left\{\gamma \in \Lambda \mid E(\gamma) < a \text{ and } \gamma|_{[i/k,(i+1)/k]} \text{ is an } F\text{-geodesic for } i = 0, \ldots, k-1\right\}$$

is a $(k \cdot \dim M)$-dimensional submanifold of $\Lambda$ consisting of closed geodesic polygons with $k$ vertices. The set $\Lambda(k, a)$ is invariant under the subgroup $\mathbf{Z}_k$ of $S^1$. The important point is that the energy functional $E$ is smooth on the open and dense subset of $\Lambda(k, a)$ that consists of all polygons for which no two consecutive vertices coincide. Moreover the closed geodesics in $\Lambda^{a-} = \{\gamma \in \Lambda \mid E(\gamma) < a\}$ are precisely the critical points of $E|_{\Lambda(k,a)}$, and for every closed geodesic $c \in \Lambda(k, a)$ the index of $(E|_{\Lambda(k,a)})''(c)$ equals $i(c)$ and the null space of $(E|_{\Lambda(k,a)})''(c)$ coincides with the nullspace of $E''(c)$, cf. [37, p. 51]. Finally, there exists a $\mathbf{Z}_k$-equivariant, energy non-increasing deformation retraction

$$r : \Lambda^{a-} \times [0, 1] \to \Lambda^{a-} \tag{3.8}$$

of $\Lambda^{a-}$ to $\Lambda(k, a)$, cf. [37, Section 6.2]. It is defined by

$$\begin{aligned} r(\gamma, u)|_{[i/k,(i+u)/k]} &= \text{ the minimal geodesic from } \gamma(i/k) \text{ to } \gamma((i+u)/k), \\ r(\gamma, u)|_{[(i+u)/k,(i+1)/k]} &= \gamma|_{[(i+u)/k,(i+1)/k]}, \end{aligned}$$

for $\gamma \in \Lambda^{a-}$, $u \in [0, 1]$ and $i = 0, \ldots, k-1$.

Throughout the paper, we will assume that each closed geodesic $c \in \Lambda$ satisfies the following isolation condition:

*(Iso) For all $m \geq 1$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of $E$.*

Since our aim is to prove the existence of more than one prime closed geodesic for every Finsler metric on $S^2$, the condition (Iso) does not restrict generality.

If $c$ has multiplicity $m$, then the subgroup

$$\mathbf{Z}_m = \left\{ \frac{k}{m} \;\middle|\; 0 \leq k < m \right\}$$

of $S^1$ acts on $C_*(E, c)$. Quite generally, if $\mathbf{Z}_m$ acts on a set $H$, we denote by $H^{\mathbf{Z}_m}$ the set of elements of $H$ fixed by $\mathbf{Z}_m$.

Our first aim is to prove

**Proposition 3.2** *If $c$ is a closed geodesic of multiplicity $m$ satisfying (Iso), then we have the following natural isomorphisms for all $q \in \mathbf{Z}$:*

$$C_q(E, S^1 \cdot c) = C_{q-1}(E, c)^{\mathbf{Z}_m} \oplus C_q(E, c)^{\mathbf{Z}_m}, \tag{3.9}$$

$$\overline{C}_q(E, c) = C_q(E, c)^{\mathbf{Z}_m}. \tag{3.10}$$

The following lemmas will help prove Proposition 3.2.

**Lemma 3.3** *Let $\Lambda(k, a) \subseteq \Lambda$ be a finite-dimensional approximation containing a closed geodesic $c$. Let $D \subseteq \Lambda(k, a)$ be a hypersurface transverse to $S^1 \cdot c$ at $c \in D$, and set $D^- = D \cap \Lambda(c)$. Then the inclusion $D^- \cup \{c\} \to \Lambda(c) \cup \{c\}$ induces an isomorphism*

$$H_*(D^- \cup \{c\}, D^-) = C_*(E, c).$$

*Proof* Using the deformation retraction $r_1 = r(\cdot, 1)$ defined in (3.8) we see that the inclusion induces an isomorphism

$$H_* \left((\Lambda(k, a) \cap \Lambda(c)) \cup \{c\}, \Lambda(k, a) \cap \Lambda(c)\right) = C_*(E, c). \tag{3.11}$$

Next, we intend to deform a neighborhood $V \subseteq \Lambda(k, a)$ of $c$ into $D$ without increasing energy. The energy non-increasing smooth map $G : \Lambda(k, a) \times S^1 \to \Lambda(k, a)$ defined by

$$G(\gamma, s) = r_1(s \cdot \gamma)$$

is a submersion in a neighborhood $U$ of $(c, 0)$ in $\Lambda(k, a) \times S^1$.

Since $\frac{\partial G}{\partial s}(c, 0)$ is tangent to $S^1 \cdot c$, while $D$ is transverse to $S^1 \cdot c$, we can find an open neighborhood $V$ of $c$ in $\Lambda(k, a)$ and $\epsilon > 0$ such that a smooth function $\sigma : V \to (-\epsilon, \epsilon)$ is uniquely defined by

$$G(\gamma, \sigma(\gamma)) \in D.$$

Now, we define $h : V \times [0, 1] \to \Lambda(k, a)$ by

$$h(\gamma, t) = G(\gamma, t\sigma(\gamma)) = r_1((t\sigma(\gamma)) \cdot \gamma).$$

Then we have $h_0 = \mathrm{id}_V$, $h_1(V) \subseteq D$, $h(\gamma, t) = \gamma$ for every $\gamma \in D \cap V$, $t \in [0, 1]$, and $E \circ h \leq E$. Using the homotopy $h$ and excision one can see that the inclusion $D^- \cup \{c\} \to (\Lambda(k, a) \cap \Lambda(c)) \cup \{c\}$ induces an isomorphism

$$H_*(D^- \cup \{c\}, D^-) = H_*((\Lambda(k, a) \cap \Lambda(c)) \cup \{c\}, \Lambda(k, a) \cap \Lambda(c)). \tag{3.12}$$

Now (3.11) and (3.12) imply our claim. □

We need the following variants of Lemma 3.3 that involve the isotropy group of $c$.

**Lemma 3.4** *Let $c$ be a closed geodesic of multiplicity $m \geq 2$ and $\Lambda(j, a) \subseteq \Lambda$ a finite-dimensional approximation containing $c$ and such that $m$ divides $j$. Let $D \subseteq \Lambda(j, a)$ be a $\mathbf{Z}_m$-invariant hypersurface transverse to $S^1 \cdot c$ at $c \in D$, and set $D^- = D \cap \Lambda(c)$. Then the inclusion $D^- \cup \{c\}/\mathbf{Z}_m \to \Lambda(c) \cup \{c\}/\mathbf{Z}_m$ induces an isomorphism*

$$H_*(D^- \cup \{c\}/\mathbf{Z}_m, D^-/\mathbf{Z}_m) \to H_*(\Lambda(c) \cup \{c\}/\mathbf{Z}_m, \Lambda(c)/\mathbf{Z}_m).$$

*Moreover, let $\Gamma \simeq \mathbf{Z}_m$ act on $(\Lambda(c) \cup \{c\}) \times S^1$ by*

$$\frac{k}{m}(\gamma, s) = \left(\frac{k}{m} \cdot \gamma, s - \frac{k}{m}\right), \quad \textit{for } \; k = 0, \ldots, m - 1. \tag{3.13}$$

*Then the inclusion $((D^- \cup \{c\}) \times S^1)/\Gamma \to (\Lambda(c) \times S^1)/\Gamma$ induces an isomorphism*

$$H_*(((D^- \cup \{c\}) \times S^1)/\Gamma, (D^- \times S^1)/\Gamma) \to H_*(((\Lambda(c) \cup \{c\}) \times S^1)/\Gamma, (\Lambda(c) \times S^1)/\Gamma).$$

*Remark* If one applies the exponential map of $\Lambda(j,a)$ to a small neighborhood of the origin in the normal space at $c$ of $S^1 \cdot c$ in $\Lambda(j,a)$, then one obtains a hypersurface $D$ satisfying the assumptions made in Lemma 3.4.

*Proof* All the constructions in the proof of Lemma 3.3 are $\mathbf{Z}_m$-equivariant. In particular, one can choose the neighborhood $V$ of $c$ in $\Lambda(j,a)$ to be $\mathbf{Z}_m$-invariant, and then the homotopy $h$ satisfies $h(\frac{k}{m}\cdot\gamma,t)=\frac{k}{m}\cdot h(\gamma,t)$ for all $k=0,\dots,m-1$, $\gamma\in V$ and $t\in[0,1]$. This implies the first statement of Lemma 3.4. The proof for the second statement is analogous. Here one defines a homotopy $\tilde{h}:((V\times S^1)/\Gamma)\times[0,1]\to(\Lambda(k,a)\times S^1)/\Gamma$ by

$$\tilde{h}([\gamma,s],t)=[h(\gamma,t),s],$$

where the square brackets denote the $\Gamma$-orbits. □

**Lemma 3.5** *Assume that the closed geodesic $c\in\Lambda(k,a)$ satisfies (Iso) and $D\subseteq\Lambda(k,a)$ is a hypersurface transverse to $S^1\cdot c$ at $c\in D$. Then $c$ is an isolated critical point of $E|_D$.*

*Proof* It suffices to prove that for $\gamma\in D\setminus\{c\}$ close to $c$, the hyperplane $T_\gamma D$ is not contained in the kernel of $E'(\gamma)$. Consider the curve $\Gamma_\gamma: S^1\to\Lambda(k,a)$ defined by

$$\Gamma_\gamma(s)=r_1(s\cdot\gamma),$$

where $r_1$ denotes the retraction $\Lambda^{a-}\to\Lambda(k,a)$ used in the proof of Lemma 3.3. Since we have $E\circ\Gamma_\gamma(s)\le E\circ\Gamma_\gamma(1)$ for all $s\in S^1$, we see that

$$\Gamma'_\gamma(1)\in\ker E'(\Gamma_\gamma(1))=\ker E'(\gamma). \tag{3.14}$$

Since $\Gamma'_c(1)$ is tangent to $S^1\cdot c$ and $D$ is transverse to $S^1\cdot c$ in $c$, we have $\Gamma'_c(1)\notin T_cD$. By continuity we see that

$$\Gamma'_\gamma(1)\notin T_\gamma D. \tag{3.15}$$

if $\gamma\in D$ is close to $c$. Since $S^1\cdot c$ is an isolated critical orbit, all $\gamma\in D\setminus\{c\}$ close to $c$ are regular points of $E|_{\Lambda(k,a)}$. Therefore both $\ker E'(\gamma)\cap T_\gamma\Lambda(k,a)$ and $T_\gamma D$ are codimension 1 subspaces in $T_\gamma\Lambda(k,a)$. By (3.14) and (3.15), they are different. Thus there exists $\xi\in T_\gamma D$ such that $E'(\gamma)(\xi)\neq 0$. □

**Lemma 3.6** *Let $D$ be a finite-dimensional Riemannian manifold on which $\mathbf{Z}_m\subseteq S^1$ acts by isometries. Let $E:D\to\mathbf{R}$ be smooth and $\mathbf{Z}_m$-invariant, and suppose $c\in D$ is the only critical point of $E$. Set $D^-=E^{-1}((-\infty,E(c)))$. Then the transfer homomorphism*

$$H_*(D^-\cup\{c\}/\mathbf{Z}_m,D^-/\mathbf{Z}_m)\to H_*(D^-\cup\{c\},D^-)^{\mathbf{Z}_m}$$

*is an isomorphism.*

*Proof* This depends on the fact that we use homology with rational coefficients. If we would use Čech homology instead of singular homology, Lemma 3.6 would directly follow from [9, Theorem III.7.2]. Since Čech and singular homology coincide for triangulable pairs, it is sufficient to show that, after appropriate excision, $(D^- \cup \{c\}/\mathbf{Z}_m, D^-/\mathbf{Z}_m)$ is homotopy equivalent to a triangulable pair. Note that, by [10, Theorem I.7.8], we can perturb $E$ in an arbitrarily small neighborhood of $c$ to a $\mathbf{Z}_m$-invariant function with only non-degenerate critical points. Then $\mathbf{Z}_m$-equivariant Morse theory, cf. [42], can be applied to finish the proof. A different argument proving Lemma 3.6 is contained in [37, Section 6.3]. Here, Rademacher uses relative $\mathbf{Z}_m$-CW-complexes. □

*Proof of Proposition 3.2* Choose a tubular neighborhood $W$ of the circle $S^1 \cdot c$ in $\Lambda$ with fibers $W_{s\cdot c} = s \cdot W_c$ over $s \cdot c \in S^1 \cdot c$, cf. [24, Lemma 2.2.8]. As on pp. 502–503 of [15] we see that the map

$$W_c \times S^1 \to W, \ (\gamma, s) \mapsto s \cdot \gamma,$$

is a normal covering with group of covering transformations $\Gamma \simeq \mathbf{Z}_m$ operating by (3.13). This together with excision provides an isomorphism

$$H_*(((W_c^- \cup \{c\}) \times S^1)/\Gamma, (W_c^- \times S^1)/\Gamma) = C_*(E, S^1 \cdot c). \tag{3.16}$$

Here and below we set $W^- = W \cap \Lambda(c)$ and $W_c^- = W_c \cap \Lambda(c)$. Now choose a finite-dimensional approximation $\Lambda(j, a) \subseteq \Lambda$ and a hypersurface $D \subseteq W_c$ in $\Lambda(j, a)$ as in Lemma 3.4. Since some neighborhood of $c$ in $\Lambda$ is $E$-equivariantly homeomorphic to $W_c \times (-\epsilon, \epsilon)$ with $\epsilon \in (0, \frac{1}{2m})$, we can use excision to obtain an isomorphism

$$\begin{aligned} &H_*\left(\left((W_c^- \cup \{c\}) \times S^1\right)/\Gamma, (W_c^- \times S^1)/\Gamma\right) \\ &\quad = H_*\left(\left((\Lambda(c) \cup \{c\}) \times S^1\right)/\Gamma, (\Lambda(c) \times S^1)/\Gamma\right). \end{aligned} \tag{3.17}$$

Then (3.16), (3.17) and Lemma 3.4 provide an isomorphism

$$H_*\left(\left((D^- \cup \{c\}) \times S^1\right)/\Gamma, (D^- \times S^1)/\Gamma\right) = C_*(E, S^1 \cdot c). \tag{3.18}$$

Now the proof of Lemma 2.6 shows that the transfer is an isomorphism

$$H_*((D^- \cup \{c\}) \times S^1, D^- \times S^1)^\Gamma = H_*(((D^- \cup \{c\}) \times S^1)/\Gamma, (D^- \times S^1)/\Gamma) \tag{3.19}$$

Using the Künneth formula we see that (3.18) and (3.19) provide an isomorphism

$$C_q(E, S^1 \cdot c) = H_{q-1}(D^- \cup \{c\}, D^-)^{\mathbf{Z}_m} \oplus H_q(D^- \cup \{c\}, D^-)^{\mathbf{Z}_m}$$

for all $q \in \mathbf{Z}$. By Lemma 3.3 this implies (3.9).

Finally, the pair $(W^- \cup S^1 \cdot c/S^1, W^-/S^1)$ is homeomorphic to the pair $(W_c^- \cup \{c\}/\mathbf{Z}_m, W_c^-/\mathbf{Z}_m)$ obviously. Using Lemma 2.4 and similar arguments as before, we obtain an isomorphism

$$H_*(D^- \cup \{c\}/\mathbf{Z}_m, D^-/\mathbf{Z}_m) = \overline{C}_*(E, c).$$

Now we can apply Lemmas 3.3 and 3.6 to obtain (3.10). □

We will now apply the results by Gromoll and Meyer [14] to a given closed geodesic $c$ satisfying (Iso). If $m = m(c)$ is the multiplicity of $c$, we choose a finite-dimensional approximation $\Lambda(k, a) \subseteq \Lambda$ containing $c$ such that $m$ divides $k$. Then the isotropy subgroup $\mathbf{Z}_m \subseteq S^1$ of $c$ acts on $\Lambda(k, a)$ by isometries. Let $D$ be a $\mathbf{Z}_m$-invariant local hypersurface transverse to $S^1 \cdot c$ at $c \in D$. Such $D$ can be obtained by applying the exponential map of $\Lambda(k, a)$ at $c$ to the normal space to $S^1 \cdot c$ at $c$. We let

$$T_c D = V_+ \oplus V_- \oplus V_0 \tag{3.20}$$

denote the orthogonal decomposition of $T_c D$ into the positive, negative and null eigenspace of the endomorphism of $T_c D$ associated to $(E|_D)''(c)$ by the Riemannian metric. In particular, we have $\dim V_- = i(c)$ and $\dim V_0 = \nu(c)$. According to [14, Lemma 1], there exist balls $B_+ \subseteq V_+$, $B_- \subseteq V_-$, and $B_0 \subseteq V_0$ centered at the origins and a diffeomorphism

$$\psi : B_+ \times B_- \times B_0 \to \psi(B_+ \times B_- \times B_0) \subseteq D \tag{3.21}$$

such that $\psi(0, 0, 0) = c$, $\psi_{*(0,0,0)}$ preserves the splitting (3.20), and

$$E \circ \psi(x_+, x_-, x_0) = |x_+|^2 - |x_-|^2 + f(x_0), \tag{3.22}$$

where $f : B_0 \to \mathbf{R}$ satisfies $f'(0) = 0$ and $f''(0) = 0$. Since the $\mathbf{Z}_m$-action is isometric and preserves $E$, the differential $(\frac{1}{m}|_D)_{*c}$ preserves the splitting (3.20). It follows from the construction of $\psi$ that $\psi$ is equivariant with respect to the $\mathbf{Z}_m$-action, i.e.,

$$\frac{1}{m} \circ \psi = \psi \circ \left(\frac{1}{m}|_D\right)_{*c}, \tag{3.23}$$

cf. [15, p. 501]. As usual, we call

$$U = \{\psi(0, x_-, 0) \mid x_- \in B_-\}$$

a local negative disk at $c$, and

$$N = \{\psi(0, 0, x_0) \mid x_0 \in B_0\}$$

a local characteristic manifold at $c$. By (3.23), local negative disks and local characteristic manifolds are $\mathbf{Z}_m$-invariant.

It follows from Lemma 3.5 and (3.22) that $c$ is an isolated critical point of $E|_N$. We set $N^- = N \cap \Lambda(c)$, $U^- = U \cap \Lambda(c) = U \backslash \{c\}$ and $D^- = D \cap \Lambda(c)$. Using (3.22) and the fact that $c$ is an isolated critical point of $E|_N$ and the Künneth formula one concludes

$$H_*(D^- \cup \{c\}, D^-) = H_*(U^- \cup \{c\}, U^-) \otimes H_*(N^- \cup \{c\}, N^-), \tag{3.24}$$

where

$$H_q(U^- \cup \{c\}, U^-) = H_q(U, U \backslash \{c\}) = \begin{cases} \mathbf{Q}, & \text{if } q = i(c), \\ 0, & \text{otherwise,} \end{cases} \tag{3.25}$$

cf. [37, Lemma 6.4] and its proof, or [14, Lemma 6].

Using Proposition 3.2, Lemma 3.3, (3.24) and (3.25), we obtain the following version of the Gromoll-Meyer Shifting Lemma:

**Proposition 3.7** *Suppose $c$ is a closed geodesic of multiplicity $m(c) = m$ satisfying (Iso). Let $U$ be a local negative disk at $c$ and let $N$ be a local characteristic manifold at $c$. Then for $q \in \mathbf{Z}$ there holds*

$$\begin{aligned} C_q(E, S^1 \cdot c) = &\left(H_{i(c)}(U^- \cup \{c\}, U^-) \otimes H_{q-i(c)}(N^- \cup \{c\}, N^-)\right)^{\mathbf{Z}_m} \\ &\oplus \left(H_{i(c)}(U^- \cup \{c\}, U^-) \otimes H_{q-1-i(c)}(N^- \cup \{c\}, N^-)\right)^{\mathbf{Z}_m}. \end{aligned}$$

In order to obtain a more explicit version of the formula in Proposition 3.7 one needs to know if a generator of the $\mathbf{Z}_m$-action on $U$ reverses orientation or not. This has been investigated by Svarc [40], see also [23, 24], Lemma 4.1.4, and [37, Sect. 6.3].

We introduce the following notation. If the group $\mathbf{Z}_m$ acts linearly on a vector space $H$ and if $T$ is a generator of $\mathbf{Z}_m$, we let $H^{\mathbf{Z}_m,1} = H^{\mathbf{Z}_m}$ denote the eigenspace of $T$ corresponding to 1, while $H^{\mathbf{Z}_m,-1}$ denotes the eigenspace of $T$ corresponding to $-1$. This is independent of the choice of the generator $T$ of $\mathbf{Z}_m$. If $m$ is odd, then $H^{\mathbf{Z}_m,-1} = \{0\}$.

**Proposition 3.8** *Let $c$ be a prime closed geodesic satisfying (Iso) and let $m \in \mathbf{N}$.*

(i) *If $\nu(c^m) = 0$, then*

$$C_q(E, S^1 \cdot c^m) = \begin{cases} \mathbf{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbf{Z}, \textit{ and } q \in \{i(c^m), i(c^m) + 1\} \\ 0, & \textit{otherwise.} \end{cases}$$

(ii) *If $\nu(c^m) > 0$, let $N_{c^m}$ be a local characteristic manifold at $c^m$ and $N_{c^m}^- = N_{c^m} \cap \Lambda(c^m)$. We set $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$. Then we have*

$$\begin{aligned} C_q(E, S^1 \cdot c^m) = &H_{q-i(c^m)}(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{\mathbf{Z}_m, \epsilon(c^m)} \\ &\oplus H_{q-1-i(c^m)}(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{\mathbf{Z}_m, \epsilon(c^m)}. \end{aligned}$$

*Proof* Let $U$ denote a local negative disk at $c^m$. The question of whether a generator $T$ of the $\mathbf{Z}_m$-action on $U$ acts as multiplication by $+1$ or $-1$ on $H_{i(c^m)}(U, U\setminus\{c^m\}) = \mathbf{Q}$ reduces to the question whether $\det(D(T|_U)_{c^m})$ is positive or negative. Let $a_+$ and $a_-$ denote the dimensions of the $(+1)$-eigenspace and the $(-1)$-eigenspace of $D(T|_U)_{c^m}$ respectively. Since $T$ is an isometry we see that

$$a_+ + a_- = \dim(U) = i(c^m) \mod 2,$$

and

$$\det(D(T|_U)_{c^m}) = (-1)^{a_-}.$$

Since $DT_{c^m}(\xi) = \xi$ for some $\xi \in T_{c^m}\Lambda$ if and only if $\xi = D\phi_m(\tilde{\xi})$ for some $\tilde{\xi} \in T_c\Lambda$, we conclude from (3.6) that $a_+ = i(c)$ and hence

$$a_- = i(c^m) - i(c) \mod 2.$$

This implies

$$\det(D(T|_U)_{c^m}) = \epsilon(c^m).$$

Now our claim follows from Proposition 3.7. □

**Definition 3.9** (i) Suppose $c$ is a closed geodesic such that $\nu(c^n) > 0$ for some $n \in \mathbf{N}$. Then we set

$$n_c = n(c) = \min\{n \in \mathbf{N} \mid \nu(c^n) > 0\}.$$

(ii) Suppose $c$ is a closed geodesic of multiplicity $m(c) = m$ satisfying (Iso). If $N$ is a local characteristic manifold at $c$, $N^- = N \cap \Lambda(c)$ and $j \in \mathbf{N} \cup \{0\}$, we define

$$k_j(c) = \dim H_j(N^- \cup \{c\}, N^-),$$
$$\hat{k}_j(c) = \dim H_j(N^- \cup \{c\}, N^-)^{\mathbf{Z}_m}.$$

Note that Lemma 3.3 and (3.24) imply that the numbers $k_j(c)$ and $\hat{k}_j(c)$ are independent of the choice of $N$. Moreover, we obviously have

$$0 \le \hat{k}_j(c) \le k_j(c).$$

Note that the finiteness of $k_j(c)$ follows from Lemma 2 of [14]. Since $\mathbf{Z}_m$ fixes $c$, we obviously have

$$k_0(c) = \hat{k}_0(c). \tag{3.26}$$

Finally, if $c$ is non-degenerate, i.e., if $\nu(c) = 0$, then $k_0(c) = \hat{k}_0(c) = 1$, while $k_j(c) = 0$ for $j > 0$.

The following facts will be useful.

**Lemma 3.10** *Let $c$ be a closed geodesic satisfying (Iso), and let $N$ be a local characteristic manifold at $c$. Then the following is true.*

(i) *The closed geodesic $c$ is a strict local minimum of $E|_N$ if and only if $k_0(c) \neq 0$. In particular, $k_0(c) \neq 0$ implies $k_0(c) = 1$ and $k_j(c) = 0$ for $j > 0$.*
(ii) *The closed geodesic $c$ is a strict local maximum of $E|_N$ if and only if $k_{\nu(c)}(c) \neq 0$. In particular, $k_{\nu(c)}(c) \neq 0$ implies $k_{\nu(c)}(c) = 1$ and $k_j(c) = 0$ for $j \neq \nu(c)$.*

*Remark* Note that $c$ is an isolated critical point of $E|_N$. Hence, if $c$ is a local minimum of $E|_N$, then this is strictly so, and similarly for the condition "local maximum".

*Proof* (i) Assume $k_0(c) \neq 0$, i.e., $H_0(N^- \cup \{c\}, N^-) \neq 0$. Since a connected open neighborhood $V$ of $c$ in $N$ can be continuously deformed into $N^- \cup \{c\}$ in an energy non-increasing manner, cf. [10, Theorem I.3.2], we conclude that $V^- = V \cap \Lambda(c) = \emptyset$. Hence $c$ is a strict local minimum of $E|_N$.

(ii) This is proved in [17, p. 256]. □

We now come to Gromoll-Meyer's crucial result on the type numbers $k_j(c^m)$ of an iterated prime closed geodesic $c$, cf. [15, Theorem 3]. Such a study for Lagrangian systems was carried out in [30] (cf. also Section 14.3 of [31]). We obtain a similar result for the dimensions $\hat{k}_j(c^m)$ of the $\mathbf{Z}_m$-invariant part too.

**Theorem 3.11** *Let $c$ be a prime closed geodesic in a Finsler manifold. Suppose $c$ satisfies (Iso), $m$, $n$, $p$ are integers and $m = np$. If the nullities of $c^m$ and $c^n$ satisfy*

$$\nu(c^m) = \nu(c^n),$$

*then*

$$k_j(c^m) = k_j(c^n) \quad \text{and} \quad \hat{k}_j(c^m) = \hat{k}_j(c^n)$$

*for all $j \in \mathbf{N} \cup \{0\}$.*

*Proof* We choose finite-dimensional approximations $\Lambda(k, a)$ containing $c^n$ and $\Lambda(kp, p^2a)$ containing $c^m$ and a characteristic manifold $N \subseteq \Lambda(k, a)$ at $c^n$. Note that the iteration map $\phi_p$ defined by (3.3) maps $\Lambda(k, a)$ diffeomorphically to a submanifold of $\Lambda(kp, p^2a)$. Hence $\phi_p(N)$ is a submanifold of $\Lambda(kp, p^2a)$ transverse to $S^1 \cdot c^m$ whose tangent space at $c^m$ is contained in the null space of $E''(c^m)$, cf. Lemma 3.1. Note moreover that $\dim \phi_p(N) = \dim N = \nu(c^n)$ and $\nu(c^n) = \nu(c^m)$ by assumption. Arguing as in the proof of Theorem 3 in [15], we can now invoke Lemma 7 from [14] to conclude that $\phi_p(N)$ is a characteristic manifold at $c^m$. Since $E \circ \phi_p = p^2 E$, this implies that $k_j(c^m) = k_j(c^n)$ for all $j$. If $T_{\frac{1}{n}}$ and $T_{\frac{1}{m}}$ denote the actions of $\frac{1}{n}$ and $\frac{1}{m}$ on $N$ and $\phi_p(N)$, respectively, then

$$T_{\frac{1}{m}} \circ \phi_p = \phi_p \circ T_{\frac{1}{n}},$$

cf. [37, p. 67]. Hence

$$(\phi_p)_* : H_*(N^- \cup \{c^n\}, N^-) \to H_*(\phi_p(N)^- \cup \{c^m\}, \phi_p(N)^-)$$

is an isomorphism conjugating generators of the $\mathbf{Z}_n$-action on $H_*(N^- \cup \{c^n\}, N^-)$ and the $\mathbf{Z}_m$-action on $H_*(\phi_p(N)^- \cup \{c^m\}, \phi_p(N)^-)$. This implies $\hat{k}_j(c^n) = \hat{k}_j(c^m)$ for all $j$. □

The following result on the critical modules $\overline{C}_*(E, c)$ in the quotient $\overline{\Lambda} = \Lambda/S^1$, cf. [37, Satz 6.11], will be used in Sect. 5.

**Proposition 3.12** *Let $c$ be a prime closed geodesic satisfying $(Iso)$ and let $m \in \mathbf{N}$, $q \in \mathbf{N} \cup \{0\}$. Let $N$ be a characteristic manifold at $c^m$, $N^- = N \cap \Lambda(c^m)$. Then we have*

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N^- \cup \{c^m\}, N^-)^{\mathbf{Z}_m, \epsilon(c^m)}.$$

*Proof* Using (3.10), Lemma 3.3, and (3.24) we conclude that

$$\overline{C}_q(E, c^m) = \left(H_{i(c^m)}(U^- \cup \{c^m\}, U^-) \otimes H_{q-i(c^m)}(N^- \cup \{c^m\}, N^-)\right)^{\mathbf{Z}_m}, \quad (3.27)$$

if $U$ is a local negative disk at $c^m$. Since generators of $\mathbf{Z}_m$ act on $H_{i(c^m)}(U^- \cup \{c^m\}, U^-) = \mathbf{Q}$ through multiplication by $\epsilon(c^m) = (-1)^{i(c^m)-i(c)}$, cf. the proof of Proposition 3.8, our claim follows from (3.27). □

In order to relate the critical $\mathbf{Q}$-modules $C_*(E, S^1 \cdot c)$ of closed geodesics $c$ to the homology of the loop space $\Lambda$, we will use the following fact.

**Proposition 3.13** *Suppose $u \in (a, b)$ is the only critical value of $E$ in the interval $[a, b]$, and the critical set*

$$C = \{c \mid c \text{ is a closed geodesic with } E(c) = u\}$$

*is the disjoint union of finitely many critical orbits*

$$C = \bigcup_{i=1}^{q} S^1 \cdot c_i.$$

*Then there is an isomorphism*

$$\bigoplus_{i=1}^{q} C_*(E, S^1 \cdot c_i) = H_*(\Lambda^b, \Lambda^a). \quad (3.28)$$

*Proof* We choose a finite-dimensional approximation $\tilde{\Lambda} = \Lambda(\tilde{k}, \tilde{a})$ with $\tilde{a} > b$, and we set $\tilde{\Lambda}^b := \Lambda^b \cap \tilde{\Lambda}$, $\tilde{E} = E|_{\tilde{\Lambda}}$ etc. Using the energy non-increasing deformation retraction from $\Lambda^{\tilde{a}-}$ onto $\tilde{\Lambda}$, one sees that it suffices to prove

$$\bigoplus_{i=1}^{q} C_*(\tilde{E}, S^1 \cdot c_i) = H_*(\tilde{\Lambda}^b, \tilde{\Lambda}^a), \quad (3.29)$$

where $C_*(\tilde{E}, S^1 \cdot c_i) = H_*(\tilde{\Lambda}^{u-} \cup S^1 \cdot c_i, \tilde{\Lambda}^{u-})$.

Note that

$$H_*(\tilde{\Lambda}^{u-} \cup C, \tilde{\Lambda}^{u-}) = \bigoplus_{i=1}^{q} C_*(\tilde{E}, S^1 \cdot c_i). \tag{3.30}$$

Since there are no critical values of $\tilde{E}$ in the interval $[a, u)$, the flow of -grad $\tilde{E}$ induces a strong deformation retraction of $\tilde{\Lambda}^{u-}$ onto $\tilde{\Lambda}^a$. This implies

$$H_*(\tilde{\Lambda}^b, \tilde{\Lambda}^a) = H_*(\tilde{\Lambda}^b, \tilde{\Lambda}^{u-}).$$

Next we choose disjoint tubular neighborhoods $W_i \subseteq \tilde{\Lambda}^b$ of the critical orbits $S^1 \cdot c_i$. By Lemma 3.5 we can assume that the orthogonal projection -grad $\tilde{E}^\top$ of -grad $\tilde{E}$ to the tangent spaces of the fibers of the tubular neighborhoods vanishes only on the critical set $C$.

We choose a smooth function $\lambda : \tilde{\Lambda}^b \to [0, 1]$ with support in $\cup_{i=1}^q W_i$ and such that $\lambda = 1$ holds in a neighborhood of $C$. Now we consider the vector field

$$X = (1 - \lambda)(-\text{grad } \tilde{E}) + \lambda(-\text{grad } \tilde{E}^\top)$$

on $\tilde{\Lambda}^b$. Since the restrictions of $\tilde{E}$ to the fibers of the tubular neighborhoods have only one critical point, one easily sees that a flow line of $X$ starting in $\tilde{\Lambda}^b \setminus \tilde{\Lambda}^{u-}$ either reaches $\tilde{\Lambda}^{u-}$ in finite time or converges to some single point in $C \subset \tilde{\Lambda}^b$. This allows us to define a homotopy

$$H : \tilde{\Lambda}^b \times [0, 1] \to \tilde{\Lambda}^b$$

such that

$$H(\tilde{\Lambda}^{u-} \times [0, 1]) \subseteq \tilde{\Lambda}^{u-},$$

and $H_1 = H(\cdot, 1)$ satisfies

$$H_1(\tilde{\Lambda}^b) \subseteq \tilde{\Lambda}^{u-} \cup C,$$

see e.g. [10, Theorem I.3.2]. This implies

$$H_*(\tilde{\Lambda}^b, \tilde{\Lambda}^{u-}) = H_*(\tilde{\Lambda}^{u-} \cup C, \tilde{\Lambda}^{u-}).$$

Using this and (3.30) we conclude that (3.29) is true. □

Suppose $(M, F)$ is a compact Finsler manifold that has only $q$ prime closed geodesics $c_j$ for $1 \le j \le q$. Then the Morse type numbers $M_k$ for $k \in \mathbf{N} \cup \{0\}$ are defined by

$$M_k = \sum_{\substack{1 \le j \le q \\ m \ge 1}} \dim C_k(E, S^1 \cdot c_j^m).$$

Using Proposition 3.13 we can prove the Morse inequalities in the standard fashion, see e.g. [24, Theorem 2.4.12], or [10, Theorem I.4.3]. Let $b_k = b_k(\Lambda, \Lambda^0) = \dim H_k(\Lambda, \Lambda^0)$ denote the relative Betti numbers of the pair $(\Lambda, \Lambda^0)$ with coefficients in $\mathbf{Q}$.

**Theorem 3.14** *Let $(M, F)$ be a compact Finsler manifold with only finitely many prime closed geodesics. Then for every integer $k \geq 0$ there holds*

$$M_k \geq b_k = b_k(\Lambda, \Lambda^0) \tag{3.31}$$

*and*

$$\begin{aligned} &M_k - M_{k-1} + M_{k-2} - \cdots + (-1)^{k-1} M_1 + (-1)^k M_0 \\ &\quad \geq b_k - b_{k-1} + b_{k-2} - \cdots (-1)^{k-1} b_k + (-1)^k b_0. \end{aligned} \tag{3.32}$$

## 4 Classification of closed geodesics on $S^2$

Let $c$ be a closed geodesic on a Finsler sphere $S^2 = (S^2, F)$. Denote the linearized Poincaré map of $c$ by $P_c : \mathbf{R}^2 \to \mathbf{R}^2$. Because the index iteration formulae in [31] work for Morse index of iterated closed geodesics, the iteration formula of Morse indices of $c$ must be one of the following nine cases by Theorems 8.1.4 to 8.1.7 of [31]. Here we use the notation from Section 8.1 of [31].

*Case CG-1.* $P_c$ is conjugate to a matrix $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.

In this case, by 1° of Theorem 8.1.4 of [31], we have $i(c) = 2p - 1$ for some $p \in \mathbf{N}$, and

$$i(c^m) = 2mp - 1, \quad \nu(c^m) = 1, \quad \text{for all } m \geq 1. \tag{4.1}$$

*Case CG-2.* $P_c = I_2$, the $2 \times 2$ identity matrix.

In this case by 2° of Theorem 8.1.4 of [31], we have $i(c) = 2p - 1$ for some $p \in \mathbf{N}$, and

$$i(c^m) = 2mp - 1, \quad \nu(c^m) = 2, \quad \text{for all } m \geq 1. \tag{4.2}$$

*Case CG-3.* $P_c$ is conjugate to a matrix $\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$ for some $b > 0$.

In this case by 3° of Theorem 8.1.4 of [31], we have $i(c) = 2p$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m) = 2mp, \quad \nu(c^m) = 1, \quad \text{for all } m \geq 1. \tag{4.3}$$

*Case CG-4.* $P_c$ is conjugate to a matrix $\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$ for some $b > 0$.

In this case by 1° of Theorem 8.1.5 of [31], we have $i(c) = 2p + 1$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m)=m(2p+1)-\frac{1+(-1)^m}{2}, \quad \nu(c^m)=\frac{1+(-1)^m}{2}, \quad \text{for all } m \geq 1. \tag{4.4}$$

*Case CG-5.* $P_c = -I_2$.

In this case by 2° of Theorem 8.1.5 of [31], we have $i(c) = 2p + 1$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m)=m(2p+1)-\frac{1+(-1)^m}{2}, \quad \nu(c^m)=1+(-1)^m, \quad \text{for all } m \geq 1. \tag{4.5}$$

*Case CG-6.* $P_c$ is conjugate to a matrix $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ for some $b > 0$.

In this case by 3° of Theorem 8.1.5 of [31], we have $i(c) = 2p + 1$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m) = m(2p+1), \quad \nu(c^m) = \frac{1+(-1)^m}{2}, \quad \text{for all } m \geq 1. \tag{4.6}$$

*Case CG-7.* $P_c$ is rationally elliptic, i.e., $P_c$ is conjugate to some rotation matrix $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $\theta/\pi \in \mathbf{Q}$.

In this case by Theorem 8.1.7 of [31], we have $i(c) = 2p+1$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m) = \begin{cases} 2mp + 2[\frac{m\theta}{2\pi}] + 1, & \nu(c^m) = 0, \quad \text{if } m\theta \neq 0 \mod 2\pi, \\ 2mp + 2[\frac{m\theta}{2\pi}] - 1, & \nu(c^m) = 2, \quad \text{if } m\theta = 0 \mod 2\pi. \end{cases} \tag{4.7}$$

*Case CG-8.* $P_c$ is irrationally elliptic, i.e., $P_c$ is conjugate to some rotation matrix $R(\theta)$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $\theta/\pi \notin \mathbf{Q}$.

In this case by Theorem 8.1.7 of [31], we have $i(c) = 2p+1$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m) = 2mp + 2\left[\frac{m\theta}{2\pi}\right] + 1, \quad \nu(c^m) = 0, \quad \text{for all } m \geq 1. \tag{4.8}$$

*Case CG-9.* $P_c$ is hyperbolic, i.e., $P_c$ is conjugate to the matrix $\begin{pmatrix} b & 0 \\ 0 & 1/b \end{pmatrix}$ for some $b > 0$ or $b < 0$.

In this case, by Theorem 8.1.6 of [31], we have $i(c) = p$ for some $p \in \mathbf{N} \cup \{0\}$, and

$$i(c^m) = mp, \quad \nu(c^m) = 0, \quad \text{for all } m \geq 1. \tag{4.9}$$

It is well known that, if all iterations $c^m$ of a closed geodesic $c$ are non-degenerate, $c$ must be hyperbolic or irrationally elliptic, i.e., $P_c$ is of the class CG-8 or CG-9. In this case, $c$ is called *non-degenerate*.

*Remark 4.1* We should remind the readers that for a closed geodesic $c : \mathbf{R}/(\tau\mathbf{Z}) \to M$ on a Finsler surface $M$, the linearized Poincaré map $P_c \in \mathrm{Sp}(2)$ is given by $\begin{pmatrix} x(\tau) & y(\tau) \\ \dot{x}(\tau) & \dot{y}(\tau) \end{pmatrix}$ in Section 3.4 of [23] and [24–26]. But in the notations here as well as in [29] and [31], the matrix $P_c$ is given by $\begin{pmatrix} \dot{y}(\tau) & \dot{x}(\tau) \\ y(\tau) & x(\tau) \end{pmatrix}$.

## 5 Rationally elliptic degenerate saddle closed geodesics on $S^2$

In this section, we study a particular type of closed geodesics of class CG-7.

**Definition 5.1** Let $c$ be a prime closed geodesic on a Finsler 2-sphere that is rationally elliptic, i.e., its linearized Poincaré map $P_c$ is of class CG-7 with rotation angle $\theta_c \in (0, \pi) \cup (\pi, 2\pi)$ and $\theta_c/\pi \in \mathbf{Q}$. We set $\sigma_c = \theta_c/2\pi \in (0, 1) \cap (\mathbf{Q}\backslash\{\frac{1}{2}\})$. The closed geodesic $c$ is called a degenerate saddle if it satisfies

$$i(c) = 1 \quad \text{and} \quad k_0(c^{n_c}) = k_2(c^{n_c}) = 0, \tag{5.1}$$

where $n_c$ and $k_j(c^{n_c})$ are defined in Definition 3.9.

The following consequence of a result by Rademacher [37] will be crucial in Sect. 10. We denote by $\alpha_c$ the mean index $\hat{i}(c) \equiv \lim_{m\to\infty} \frac{i(c^m)}{m}$ of a closed geodesic $c$.

**Theorem 5.2** *Let $(S^2, F)$ be a Finsler 2-sphere and assume that there exists only one prime closed geodesic $c$ on $(S^2, F)$ and that $c$ is a rationally elliptic degenerate saddle. Then there holds*

$$\frac{n_c - 1 - \hat{k}_1(c^{n_c})}{n_c\alpha_c} = 1. \tag{5.2}$$

The number $\hat{k}_1(c^{n_c})$ is defined in Definition 3.9.

*Proof* Theorem 7.9 in [37] treats compact, simply connected Finsler manifolds $(M, F)$ with only finitely many prime closed geodesics and provides a relation between invariants of these closed geodesics and a topological invariant of $M$. A simple computation shows that this last invariant equals $-1$ in the case $M = S^2$. If $c$ is the only prime closed geodesic on $(S^2, F)$, this relation says

$$\frac{\beta_c}{\alpha_c} = -1, \tag{5.3}$$

where $\beta_c$ is the invariant of $c$ defined in [37, Satz 7.3]. The proof of Theorem 5.2 consists in expressing $\beta_c$ by $n_c$ and $\hat{k}_1(c^{n_c})$ as follows

$$\beta_c = \frac{\hat{k}_1(c^{n_c}) + 1 - n_c}{n_c}. \tag{5.4}$$

Then (5.2) is a direct consequence of (5.3) and (5.4). The invariant $\beta_c$ is defined as follows. Set

$$M_{m,j}(c) = \dim \overline{C}_j(E, c^m)$$

with $\overline{C}_j(E, c^m) = H_j(\Lambda(c^m) \cup S^1 \cdot c^m/S^1, \Lambda(c^m)/S^1)$, cf. (3.7). There exists a minimal even integer $k(c) > 0$ such that for all $m \geq 1$, $j \geq 0$

$$M_{m,j+i(c^m)}(c) = M_{m+k(c),j+i(c^{m+k(c)})}(c).$$

Then

$$\beta_c = \frac{1}{k(c)} \sum_{1 \leq m \leq k(c),\, j \geq 0} (-1)^j M_{m,j}(c). \tag{5.5}$$

Using Theorem 3.11, Proposition 3.12 and the index iteration formula (4.7) we first compute the numbers $M_{m,j}(c)$ for a rationally elliptic degenerate saddle $c$. According to (4.7) we have

$$i(c^m) = 2[m\sigma_c] + 1, \quad \nu(c^m) = 0, \quad \text{if } m \notin n\mathbf{N}, \tag{5.6}$$

$$i(c^m) = 2[m\sigma_c] - 1, \quad \nu(c^m) = 2, \quad \text{if } m \in n\mathbf{N}, \tag{5.7}$$

where $\sigma_c = \theta_c/2\pi \in (0,1) \cap (\mathbf{Q} \setminus \{\frac{1}{2}\})$ and $n = n_c \geq 3$ is the denominator of the reduced fraction $\sigma_c$.

In particular, (5.6) and (5.7) imply

$$\epsilon(c^m) = (-1)^{i(c^m)-i(c)} = 1 \tag{5.8}$$

for all $m \in \mathbf{N}$. Using Proposition 3.12 we obtain

$$M_{m,j}(c) = \begin{cases} 1 & \text{if } m \notin n\mathbf{N} \text{ and } j = i(c^m), \\ 0 & \text{if } m \notin n\mathbf{N} \text{ and } j \neq i(c^m). \end{cases} \tag{5.9}$$

In the remaining cases we can use Theorem 3.11 and Proposition 3.12 to conclude

$$M_{m,j}(c) = \begin{cases} \hat{k}_1(c^n) & \text{if } m \in n\mathbf{N} \text{ and } j = i(c^m) + 1, \\ 0 & \text{if } m \in n\mathbf{N} \text{ and } j \neq i(c^m) + 1. \end{cases} \tag{5.10}$$

In particular, we can take $k(c) = 2n$. Now $\beta_c$ can be computed from (5.5), and the result of this computation is Eq. (5.4). □

Below, we will present a generalization of Theorem 5.2 which can be used in the study of non-degenerate closed geodesics in Cases CG-8 and CG-9 too. Following Theorem 7.9 of [37], we have

**Theorem 5.3** *Let $S^2 = (S^2, F)$ be a Finsler 2-sphere with only finitely many prime closed geodesics, each of which is either non-degenerate or a rationally elliptic degenerate saddle. Denote non-degenerate prime closed geodesics on $S^2$ by $c_j$ for $1 \le j \le r$, and rationally elliptic degenerate saddle prime closed geodesics on $S^2$ by $c_j$ for $r+1 \le j \le r+a < +\infty$. We denote the mean index of $c_j$ by $\alpha_j = \hat{i}(c_j)$. Let $\gamma_j = \gamma_{c_j} \in \{\pm 1/2, \pm 1\}$ such that*

$$2\gamma_j = i(c_j^2) - i(c_j) \mod 2, \quad \text{and} \quad \gamma_j(-1)^{i(c_j)} > 0. \tag{5.11}$$

*Then there holds*

$$\sum_{j=1}^{r} \frac{\gamma_j}{\alpha_j} - \sum_{j=r+1}^{r+a} \frac{(n_j - 1 - \hat{k}_1(c_j^{n_j}))}{n_j \alpha_j} = -1, \tag{5.12}$$

*where we set $n_j = n_{c_j}$ for $r+1 \le j \le r+a$.*

*Proof* Note first that, by (5.6) and (5.7), for a rationally elliptic degenerate saddle closed geodesic $c$, we always have $\gamma_c = -1$.

Note that, for a closed geodesic $c$, its mean index satisfies either $\hat{i}(c) > 0$ or $\hat{i}(c) = 0$. When $\hat{i}(c) = 0$, there holds $i(c^m) = 0$ for all $m \ge 1$.

Therefore for a non-degenerate elliptic or a rationally elliptic degenerate saddle closed geodesic, because the rotation angle is always positive, by the iteration formula (4.7) or (4.8), its mean index is positive.

For any hyperbolic closed geodesic $c$, if $\hat{i}(c) = 0$, then $i(c^m) = 0$ for all $m \ge 1$. Note that there holds $E(c^m) > E(c^{m-1})$ for all $m \ge 2$. Because $c^m$ is non-degenerate in $\Lambda$, each $c^m$ for $m \ge 1$ must be a strict local minimum of $E$ in $\Lambda$. For every strict local minimum $c^m$ with $m \ge 2$, we have $E(c^{m-1}) < E(c^m)$. Thus a mountain pass argument yields a closed geodesic $d_m$ with a non-trivial 1-dimensional local homological critical module and $E(d_m) > E(c^m)$. This argument yields infinitely many such $d_m$. Under our assumptions the mean indices of the prime closed geodesics are positive except for the hyperbolic ones of index zero, which together with all their iterates are local minima. Hence the $d_m$ cannot be iterates of finitely many closed geodesics. This proves that hyperbolic closed geodesics of index zero do not exist under our assumptions. Note that this case can also be excluded by Corollary 2 of [6].

Therefore the denominators on the left hand side of (5.12) are all non-zero, and then (5.12) holds.

Now using our Theorem 5.2 together with the proof of Theorem 3.1 of [36] and [37], we get Theorem 5.3. □

## 6 Two theorems by N. Hingston

In certain cases one can use the existence of a degenerate closed geodesic $c$ to prove the existence of infinitely many closed geodesics. The first instance of this phenomenon was discovered in [3] in the case of vanishing mean index $\hat{i}(c) = 0$, see also [5]. The method was considerably advanced by Hingston [17] and [18] who was able to treat cases where $\hat{i}(c) > 0$. The proofs of these results are variational and do not require symmetry of the metric. Hence the results apply to general Finsler metrics. The following statement combines [17, Proposition 1], and [18], Theorem, for the case of a Finsler 2-sphere $(S^2, F)$.

**Theorem 6.1** *Let $c$ be a closed geodesic on $(S^2, F)$ that satisfies (Iso). Assume that either*

(1) $k_0(c) > 0$ *and* $i(c^m) = m(i(c) + 1) - 1$, $\nu(c^m) = \nu(c)$ *for all* $m \in \mathbf{N}$,
*or*

(2) $k_{\nu(c)}(c) > 0$ *and* $i(c^m) + \nu(c^m) = m(i(c) + \nu(c) - 1) + 1$, $\nu(c^m) = \nu(c)$ *for all* $m \in \mathbf{N}$.

*Then there exist infinitely many prime closed geodesics on* $(S^2, F)$.

To see that [17, Proposition 1], and [18], Theorem, imply Theorem 6.1, note that Lemma 3.3, formulae (3.24) and (3.25) imply $C_{i(c)}(E, c) \neq 0$ in case (1), and $C_{i(c)+\nu(c)}(E, c) \neq 0$ in case (2). So, in case (1) the hypotheses of [18], Theorem, are satisfied, while in case (2) the hypotheses of [17, Proposition 1], hold.

By Lemma 3.10 the conditions $k_0(c) > 0$ or $k_{\nu(c)}(c) > 0$, respectively, are equivalent to the fact that $c$ is a strict local minimum or a strict local maximum of $E$ restricted to a local characteristic manifold at $c$.

N. Hingston imposes the seemingly weaker assumptions $i(c^m) \geq m(i(c) + 1) - 1$ in case (1), and $i(c^m) + \nu(c^m) \leq m(i(c) + \nu(c) - 1) + 1$ in case (2). The estimates in Theorems 10.1.2 and 10.1.3 of [31], originally proved in [27], imply that these inequalities are in fact equalities.

## 7 Homology of $(\Lambda S^2, \Lambda^0 S^2)$ and first consequences

In Ziller [43] computed the $\mathbf{Z}$-homology of the free loop space $\Lambda$ of compact rank 1 symmetric spaces (with the exception of $\mathbf{R}P^n$). For $\Lambda S^2$ the table on p. 21 of [43], taken literally, does not give the correct result. However, the correct result follows easily from [43, Theorem 8], and it is explicitly stated in [44, p. 148]. Specialized to the coefficient ring $\mathbf{Q}$ one has

$$H_k(\Lambda S^2) = \mathbf{Q} \tag{7.1}$$

for all $k \in \mathbf{N} \cup \{0\}$.

Next, we solve the simple exercise to compute $H_k(\Lambda S^2, \Lambda^0 S^2)$ from (7.1). If $i : \Lambda^0 S^2 \to \Lambda S^2$ denotes inclusion and $ev : \Lambda S^2 \to S^2$ denotes the evaluation map $ev(\gamma) = \gamma(1)$, then the map $ev \circ i : \Lambda^0 S^2 \to S^2$ is a diffeomorphism. This implies that $i_* : H_*(\Lambda^0 S^2) \to H_*(\Lambda S^2)$ is one-to-one. Hence the long exact homology sequence of the pair $(\Lambda S^2, \Lambda^0 S^2)$ together with (7.1) shows that

$$H_k(\Lambda S^2, \Lambda^0 S^2) = \begin{cases} 0, & \text{if } k = 0, \text{ or } k = 2, \\ \mathbf{Q}, & \text{if } k = 1, \text{ or } k \geq 3. \end{cases} \tag{7.2}$$

From now on in the rest of this paper, we write simply $\Lambda = \Lambda S^2$ and $\Lambda^a = \Lambda^a S^2$ for $a \in \mathbf{R}$. In the following three sections we will prove Theorem 1.1 by contradiction. So we will assume the condition

*(F) There exists only one prime closed geodesic c on the given Finsler 2-sphere* $(S^2, F)$.

We mention some simple consequences of $(F)$, (7.2) and the Morse inequalities (3.32). Using Proposition 3.8 and (7.2) and the fact that $\nu(c^m) \leq 2$ for all $m \in \mathbf{N}$, we see that the sequence $i(c^m)$ is unbounded. By the index iteration formulae (4.1)–(4.9) this implies $i(c^m) \geq i(c) > 0$ for all $m \in \mathbf{N}$. Using Proposition 3.8 again, we conclude that the Morse type number $M_0$ satisfies

$$M_0 = 0. \tag{7.3}$$

Moreover, by (3.31) and (7.2) we have

$$M_1 \geq b_1(\Lambda, \Lambda^0) = 1,$$

and hence $i(c) = 1$. By (4.3) this implies:

$$\text{The only prime closed geodesic } c \text{ cannot be of type CG-3.} \tag{7.4}$$

Moreover, the integer $p$ in the iteration formulae (4.1), (4.2) and (4.4)–(4.9) satisfies:

– if $c$ is of one of the types CG-1, CG-2 or CG-9, then $p = 1$; (7.5)

– if $c$ is of one of the types CG-4, CG-5, CG-6, CG-7 or CG-8, then $p = 0$. (7.6)

## 8 Cases with eigenvalue 1

We recall that we use homology with rational coefficients. We shall use the results from Sect. 3 to compute local critical modules. We recall the numbers $k_j(c)$ and $\hat{k}_j(c)$ defined in Definition 3.9.

### 8.1 Case CG-$k$ with $k = 1$ or $k = 2$

Note that (4.1) and (4.2) imply that $i(c^m) - i(c)$ is even for every $m \in \mathbf{N}$. Moreover, (7.5) implies that $p = 1$ in formulae (4.1) and (4.2). Thus, if $c$ is of type CG-$k$ with

$k \in \{1, 2\}$, then (4.1) and (4.2) become

$$i(c^m) = 2m - 1, \quad \nu(c^m) = k \quad \text{for all} \quad m \geq 1. \tag{8.1}$$

For the closed geodesic $c$ itself we obtain from Proposition 3.8

$$C_1(E, S^1 \cdot c) = H_0(N_c^- \cup \{c\}, N_c^-)^{\mathbf{Z}_m} = \mathbf{Q}^{k_0(c)}, \tag{8.2}$$

since $\hat{k}_0(c) = k_0(c)$, cf. (3.26). Here and below we denote by $\mathbf{Q}^h$ the direct sum $\mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ of $h$ copies of $\mathbf{Q}$ for any integer $h \geq 0$.

For the iterates $c^m$ with $m \geq 2$, we have $i(c^m) \geq 3$, so that Proposition 3.8 implies

$$C_1(E, S^1 \cdot c^m) = 0 \quad \text{for} \quad m \geq 2. \tag{8.3}$$

Using the Morse inequality (3.31) and the fact that $b_1 = b_1(\Lambda, \Lambda^0) = 1$ by (7.2), and (8.2) and (8.3), we obtain

$$k_0(c) = M_1 \geq b_1 = 1. \tag{8.4}$$

Now (8.1) and (8.4) imply that the hypothesis (1) of N. Hingston's Theorem 6.1 is satisfied. Hence, in contradiction to our assumption (F), there exist infinitely many prime closed geodesics on $(S^2, F)$.

### 8.2 Case CG-3

According to (7.4) this case cannot occur.

## 9 Cases with eigenvalue $-1$

### 9.1 Case CG-$k$ with $k = 4$ or $k = 5$

In these two cases, $i(c^m) - i(c)$ is even for every $m \in \mathbf{N}$ by (4.4) or (4.5). According to (7.6), we have $p = 0$ in the formulae (4.4) and (4.5). Then (4.4) or (4.5) become

$$i(c^m) = m - \frac{1+(-1)^m}{2}, \quad \nu(c^m) = \frac{(1+(-1)^m)[(k-1)/2]}{2}, \quad \text{for all} \quad m \geq 1. \tag{9.1}$$

We will now compute the Morse type numbers $M_1$, $M_2$ and $M_3$.

Since $i(c) = 1$ and $\nu(c) = 0$, we obtain from Proposition 3.8

$$C_1(E, S^1 \cdot c) = C_2(E, S^1 \cdot c) = \mathbf{Q}, \tag{9.2}$$
$$C_3(E, S^1 \cdot c) = 0. \tag{9.3}$$

For $c^2$, we have $i(c^2) = 1 = i(c)$, $\nu(c^2) = [(k-1)/2]$, and $i(c^2) - i(c) = 0$. Then by Proposition 3.8, we obtain

$$C_1(E, S^1 \cdot c^2) = \mathbf{Q}^{\hat{k}_0(c^2)}, \tag{9.4}$$

$$C_2(E, S^1 \cdot c^2) = \mathbf{Q}^{\hat{k}_1(c^2)+\hat{k}_0(c^2)}, \tag{9.5}$$

$$C_3(E, S^1 \cdot c^2) = \mathbf{Q}^{\hat{k}_2(c^2)+\hat{k}_1(c^2)}. \tag{9.6}$$

For $c^3$, we have $i(c^3) = 3$, $\nu(c^3) = 0$, and $i(c^3) - i(c) = 2$. Thus by Proposition 3.8, we have

$$C_1(E, S^1 \cdot c^3) = C_2(E, S^1 \cdot c^3) = 0, \tag{9.7}$$

$$C_3(E, S^1 \cdot c^3) = \mathbf{Q}. \tag{9.8}$$

For $c^4$, we have $i(c^4) = 3$, $\nu(c^4) = 1$ in Case CG-4 and $\nu(c^4) = 2$ in Case CG-5. Because $i(c^4) - i(c^2) = 2$, by Proposition 3.8, in both cases CG-4 and CG-5 we have

$$C_1(E, S^1 \cdot c^4) = C_2(E, S^1 \cdot c^4) = 0, \tag{9.9}$$

$$C_3(E, S^1 \cdot c^4) = \mathbf{Q}^{\hat{k}_0(c^4)}. \tag{9.10}$$

For $c^m$ with $m \geq 5$, we have $i(c^m) \geq 4$ and hence Proposition 3.8 implies

$$C_q(E, S^1 \cdot c^m) = 0, \quad \text{for all } q \leq 3, \quad m \geq 5. \tag{9.11}$$

Thus we obtain $M_1 = 1 + \hat{k}_0(c^2)$, $M_2 = 1 + \hat{k}_0(c^2) + \hat{k}_1(c^2)$, and $M_3 = 1 + \hat{k}_0(c^4) + \hat{k}_1(c^2) + \hat{k}_2(c^2)$. Then, by (3.32), (7.2), and (7.3), we obtain

$$1 + \hat{k}_0(c^4) + \hat{k}_2(c^2) = M_3 - M_2 + M_1 \geq b_3 - b_2 + b_1 = 2. \tag{9.12}$$

Therefore, at least one of $\hat{k}_0(c^4)$ and $\hat{k}_2(c^2)$ must be positive. First suppose that $\hat{k}_0(c^4)$ is positive. Then, by (3.26) and Theorem 3.11, we have

$$k_0(c^2) = \hat{k}_0(c^2) = \hat{k}_0(c^4) > 0.$$

Now consider the closed geodesic $d = c^2$. Then, we have $k_0(d) > 0$, $i(d) = 1$, and

$$i(d^m) = i(c^{2m}) = 2m - 1 = m(i(d)+1) - 1, \ \nu(d^m) = \nu(d), \quad \text{for all } m \geq 1. \tag{9.13}$$

Therefore, by Theorem 6.1, there exist infinitely many prime closed geodesics on $(S^2, F)$.

Finally suppose that $\hat{k}_2(c^2)$ is positive. This can only happen in case CG-5 when $\nu(c^2) = 2$. Considering $d = c^2$ again we have $k_{\nu(d)}(d) > 0$, $i(d) = 1$, $\nu(d) = 2$, and for all $m \geq 1$:

$$i(d^m) = i(c^{2m}) = 2m - 1 = m(i(d) + 1) - 1, \quad \nu(d^m) = \nu(d), \tag{9.14}$$

$$i(d^m) + \nu(d^m) = 2m + 1 = m(i(d) + \nu(d) - 1) + 1. \tag{9.15}$$

Again, by Theorem 6.1, we obtain infinitely many prime closed geodesics on $(S^2, F)$.

9.2 Case CG-6

According to (7.6) we have $p = 0$ in formula (4.6), so that (4.6) becomes

$$i(c^m) = m, \quad \nu(c^m) = \frac{1 + (-1)^m}{2}, \quad \text{for all } m \geq 1. \tag{9.16}$$

Note that

$$\epsilon(c^m) = (-1)^{i(c^m) - i(c)} = \begin{cases} -1, & \text{if } m \text{ is even,} \\ +1, & \text{if } m \text{ is odd.} \end{cases} \tag{9.17}$$

Next we compute the Morse type numbers $M_1$, $M_2$ and $M_3$.

Since $i(c) = 1$ and $\nu(c) = 0$, Proposition 3.8 implies

$$\begin{aligned} &C_1(E, S^1 \cdot c) = C_2(E, S^1 \cdot c) = \mathbf{Q}, \\ &C_3(E, S^1 \cdot c) = 0. \end{aligned} \tag{9.18}$$

From (9.16) we have $i(c^2) = 2$, $\epsilon(c^2) = -1$ and $\nu(c^2) = 1$. So Proposition 3.8 and (9.17) imply

$$C_q(E, S^1 \cdot c^2) = H_{q-2}\left(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-\right)^{\mathbf{Z}_2, -1} \oplus H_{q-3}\left(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-\right)^{\mathbf{Z}_2, -1},$$

where $N_{c^2}$ denotes a local characteristic manifold at $c^2$, $N_{c^2}^- = N_{c^2} \cap \Lambda(c^2)$. We will show that $H_*(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-)^{\mathbf{Z}_2, -1} = 0$, and this will prove

$$C_*(E, S^1 \cdot c^2) = 0. \tag{9.19}$$

First note that $H_0(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-)^{\mathbf{Z}_2, -1} = 0$, since the $\mathbf{Z}_2$-action fixes $c^2$. From the assumption $H_1(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-)^{\mathbf{Z}_2, -1} \neq 0$, we conclude that $k_1(c^2) = \dim H_1(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-) > 0$. In this case the hypotheses (2) of Theorem 6.1 hold for the closed geodesic $d = c^2$: therefore we have $\nu(d) = 1$, $k_{\nu(d)}(d) > 0$ and, by (9.16),

$$i(d^m) + \nu(d^m) = 2m + 1 = m(i(d) + \nu(d) - 1) + 1, \quad \nu(d^m) = \nu(d)$$

for all $m \geq 1$. Thus we obtain infinitely many prime closed geodesics, in contradiction to our assumption $(F)$. Hence we have $H_1(N_{c^2}^- \cup \{c^2\}, N_{c^2}^-)^{\mathbf{Z}_2, -1} = 0$. Since $\dim N_{c^2} = \nu(c^2) = 1$, this completes the proof that $H_*((N_{c^2}^- \cup \{c^2\}, N_{c^2}^-)^{\mathbf{Z}_2, -1} = 0$.

Since $i(c^3) = 3$ and $\nu(c^3) = 0$, Proposition 3.8 implies

$$\begin{aligned} &C_1(E, S^1 \cdot c^3) = C_2(E, S^1 \cdot c^3) = 0 \\ &C_3(E, S^1 \cdot c^3) = \mathbf{Q}. \end{aligned} \tag{9.20}$$

Finally, if $m \geq 4$, then $i(c^m) \geq 4$, and hence by Proposition 3.8 we obtain

$$C_q(E, S^1 \cdot c^m) = 0 \quad \text{for} \;\; q \in \{1, 2, 3\}. \tag{9.21}$$

Now (9.18)–(9.21) imply $M_1 = M_2 = M_3 = 1$. From the Morse inequalities (3.32) and from (7.2), (7.3), we obtain the contradiction

$$1 = M_3 - M_2 + M_1 \geq b_3 - b_2 + b_1 = 2.$$

Hence, only one closed geodesic of type CG-6 cannot generate all the homology of $(\Lambda, \Lambda^0)$.

## 10 Case CG-7 of a rationally elliptic closed geodesic

In this section, we will derive a contradiction from the assumption (F) that a Finsler sphere $(S^2, F)$ has only one prime closed geodesic $c$ if this $c$ is of type CG-7, i.e., if the linearized Poincaré map of $c$ is conjugate to a rotation by an angle $\theta \in (0, \pi) \cup (\pi, 2\pi)$ with $\theta/\pi \in \mathbf{Q}$.

Our arguments use first N. Hingston's results, Theorem 6.1, to reduce the problem to the subcase of a rationally elliptic degenerate saddle, cf. Definition 5.1. Then using Theorem 5.2 or Theorem 5.3 which are based on H.-B. Rademacher's work and the index iteration formula (4.7), we further restrict the rotation angle $\theta$. These results allow us to show that $c$ and its iterates generate a surplus in local one-dimensional homology. A careful analysis of the situation then shows that the local 2-dimensional homology generated by the iterates of $c$ cannot destroy this surplus in one-dimensional homology. This is the final contradiction. More precisely, this contradiction is reached in the following three steps.

*Step 1* General information on the closed geodesic $c$.

We first mention some consequences of our assumptions on $c$.
We set

$$\sigma = \theta/2\pi. \tag{10.1}$$

Then $\sigma \in (0, 1) \cap (\mathbf{Q} \setminus \{1/2\})$. From (7.6) we know that the integer $p$ in (4.7) equals zero. Hence (4.7) becomes

$$i(c^m) = 2[m\sigma] + 1, \quad \nu(c^m) = 0, \quad \text{if} \;\; m\sigma \notin \mathbf{N}, \tag{10.2}$$
$$i(c^m) = 2[m\sigma] - 1, \quad \nu(c^m) = 2, \quad \text{if} \;\; m\sigma \in \mathbf{N}. \tag{10.3}$$

In particular, we have $i(c) = 1$ and the mean index $\alpha \equiv \hat{i}(c)$ satisfies $\alpha = 2\sigma$. Moreover, with $n \equiv n_c \in \mathbf{N}$ given by Definition 3.9, we have $n \geq 3$, and $k \equiv n\sigma$ satisfies $k \in \{1, \ldots, n-1\}$ and is relatively prime to $n$. Then $d = c^n$ is a degenerate closed geodesic satisfying $i(d) = 2k - 1$, $\nu(d) = 2$, $\epsilon(c^n) = (-1)^{i(c^n)-i(c)} = 1$, and

$$i(d^m) = 2km - 1 = m(i(d) + 1) - 1, \quad \nu(d^m) = 2, \tag{10.4}$$
$$i(d^m) + \nu(d^m) = 2km + 1 = m(i(d) + \nu(d) - 1) + 1, \tag{10.5}$$

for all $m \in \mathbf{N}$. Hence, the closed geodesic $d$ satisfies part of the assumptions of N. Hingston's Theorem 6.1. Since this result promises the existence of infinitely many closed geodesics on $(S^2, F)$, its additional assumptions $k_0(d) > 0$ or $k_2(d) > 0$ are both not true, i.e., $c$ is a rationally elliptic degenerate saddle in the sense of Definition 5.1. Hence our Theorem 5.2 implies the following crucial identity, which restricts the rotation angle $\theta$:

$$n - 1 - \hat{k}_1(c^n) = n\alpha, \tag{10.6}$$

where $\hat{k}_1(c^n)$ is given by Definition 3.9.

Since $\alpha = 2\sigma$ and $\sigma = k/n$ with $k \geq 1$, we obtain

$$0 \leq \hat{k}_1(c^n) = n - 1 - 2k \leq n - 3, \tag{10.7}$$

and, in particular, $2k + 1 \leq n$. Hence we have

$$2\sigma < 1, \tag{10.8}$$

i.e., $\theta < \pi$. We set

$$\tau = \max\{m \in \mathbf{N} \mid m\sigma < 1\}. \tag{10.9}$$

Because of (10.8) and $\sigma = k/n$ we have

$$2 \leq \tau \leq n - 1. \tag{10.10}$$

*Step 2* Vanishing connecting homomorphism and additive homologies among level sets.

Now we will study the one-dimensional homology generated by the closed geodesics $c, c^2, \ldots, c^\tau$. By (10.2) and (10.9) all of them are non-degenerate and of index one. We set $\kappa_0 = 0$ and

$$\kappa_m = E(c^m), \quad \text{for all } m \in \mathbf{N}.$$

There holds

$$0 = \kappa_0 < \kappa_1 < \cdots \kappa_m < \kappa_{m+1} < \cdots, \tag{10.11}$$
$$\kappa_m \to +\infty, \quad \text{as} \quad m \to +\infty. \tag{10.12}$$

Note that

$$H_0(\Lambda^{\kappa_m}, \Lambda^0) = 0, \quad \text{for all } m \in \mathbf{N}, \tag{10.13}$$

holds, since there are no closed geodesics of index zero. We recall that, for all $m \in \mathbf{N}$, we have $H_*(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) = C_*(E, S^1 \cdot c^m)$, cf. Proposition 3.13.

In [8] of 1958, Bott and Samelson established the additivity of homologies of level sets for pointed loop spaces of compact globally symmetric spaces. In [43] of 1977, Ziller established this additivity for free loop spaces of such spaces. In general $(S^2, F)$ is not a globally symmetric space. Our following result establishes also such an additivity of the homologies of level sets of the energy functional on $(\Lambda^{\kappa_\tau}, \Lambda^0)$ for Case CG-7 by a rather different method.

**Proposition 10.1** *Under the assumption (F), let c be a prime closed geodesic of type CG-7. Then*

$$H_1(\Lambda^{\kappa_\tau}, \Lambda^0) = \mathbf{Q}^\tau.$$

*Proof* In Lemma 10.2 below we will show that for every $2 \le m \le \tau$ the connecting homomorphism

$$\partial_2 : H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) \to H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0)$$

of the exact homology sequence of the triple $(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}, \Lambda^0)$ is zero. Using this and the fact that $H_0(\Lambda^{\kappa_{m-1}}, \Lambda^0) = 0$, cf. (10.13), this exact sequence splits and implies that

$$H_1(\Lambda^{\kappa_m}, \Lambda^0) = H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0) \oplus H_1(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}).$$

Since $i(c^m) - i(c) = 0$, (i) of Proposition 3.8 implies that $H_1(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) = \mathbf{Q}$ for $1 \le m \le \tau$. Hence our claim follows by induction. □

**Lemma 10.2** *Under the assumption of Proposition 10.1, for $2 \le m \le \tau$ the connecting homomorphism*

$$\partial_2 : H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) \to H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0)$$

*of the triple $(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}, \Lambda^0)$ is zero.*

*Proof* Recall that

$$i(c^m) = 1, \quad \nu(c^m) = 0, \quad \text{for } 1 \le m \le \tau. \tag{10.14}$$

So the local negative disks $U_{c^m}$ are one-dimensional, and we can assume that the m'th iteration map $\phi_m$ maps $U_c$ onto $U_{c^m}$. This implies that the $\mathbf{Z}_m$-action on $U_{c^m}$ is trivial. Using Proposition 3.8 and (10.14) we see that

$$H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) = C_2(E, S^1 \cdot c^m) = \mathbf{Q}, \quad \text{for } 1 \le m \le \tau. \tag{10.15}$$

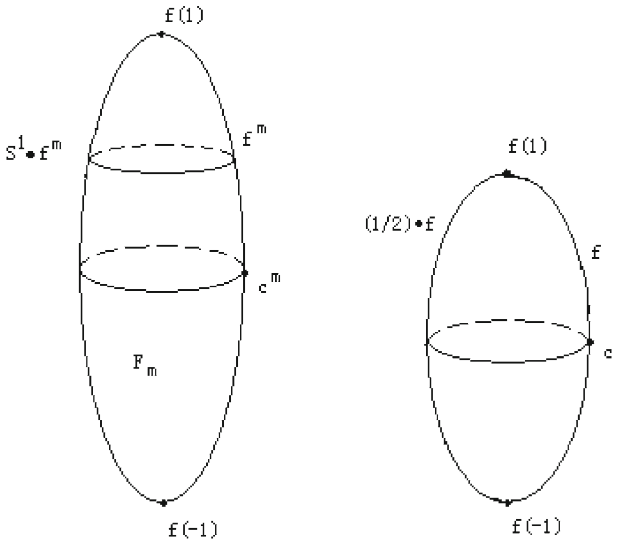


**Fig. 1** The map $F_m$

Note that a representative for a generator of $H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) = C_2(E, S^1 \cdot c^m)$ can be constructed as follows.

(i) Let $f : ([-1, 1], \{-1, 1\}) \to (\Lambda^{\kappa_1}, \Lambda^0)$ be a continuous map such that $f(0) = c$ and satisfying $0 < E(f(t)) < \kappa_1$ for $t \in (-1, 1) \setminus \{0\}$, and such that, for some $\epsilon > 0$, $f|_{[-\epsilon,\epsilon]}$ represents a generator of $H_1(U_c, U_c^-)$. Such an $f$ exists by (10.14). Then, by Theorem 3.11, for every $m \in \{1, 2, \ldots, \tau\}$, the curve $f^m := \phi_m \circ f : [0, 1] \to \Lambda^{\kappa_m}$ has the property that $f^m|_{[-\epsilon,\epsilon]}$ generates $H_1(U_{c^m}, U_{c^m}^-)$, cf. Fig. 1.

(ii) Define $F_m : (S^1 \times [-1, 1], S^1 \times \{-1, 1\}) \to (\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}})$ by $F_m(\theta, t) = \theta \cdot f^m(t)$, cf. Fig. 1. If $0 \neq h \in H_2(S^1 \times [-1, 1], S^1 \times \{-1, 1\})$ is the standard generator, then $0 \neq (F_m)_*(h) \in H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}})$, cf. the proof of Proposition 3.2. Hence $(F_m)_*(h)$ generates $H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}})$.

By our above construction, $\partial_2((F_m)_*h) = (F_m)_*(\partial_2 h)$ is the difference of the homology classes of two trivial $S^1$-orbits

$$\begin{aligned}\partial_2((F_m)_*h) &= [S^1 \cdot f^m(1)] - [S^1 \cdot f^m(-1)]\\ &= [f(1)] - [f(-1)]\\ &= 0 \quad \text{in } H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0).\end{aligned}$$

□

*Step 3* Chasing exact sequences.

Now we treat the case where $\tau < n - 1$, i.e., $k = n\sigma > (\tau + 1)\sigma > 1$. By (10.2) and (10.3) this entails $i(c^m) \geq 3$ for all $m > \tau$. Hence we have

$$H_2(\Lambda, \Lambda^{\kappa_\tau}) = 0.$$

Now the exact sequence

$$H_2(\Lambda, \Lambda^{\kappa_\tau}) \to H_1(\Lambda^{\kappa_\tau}, \Lambda^0) \to H_1(\Lambda, \Lambda^0)$$

implies that $\dim H_1(\Lambda^{\kappa_\tau}, \Lambda^0) \le \dim H_1(\Lambda, \Lambda^0)$. Since $\dim H_1(\Lambda, \Lambda^0) = 1$ by (7.2), this contradicts Proposition 10.1 and the fact that $\tau \ge 2$, cf. (10.10).

Finally, we consider the case that $\tau = n - 1$, i.e., $k = n\sigma = (\tau + 1)\sigma = 1$. Then we have $c^{\tau+1} = c^n = d$, $i(d) = 1$, $\nu(d) = 2$ and $k_0(d) = k_2(d) = 0$, since $c$ is a rationally elliptic degenerate saddle. Together with Proposition 3.8, we obtain

$$\dim H_2(\Lambda^{\kappa_{\tau+1}}, \Lambda^{\kappa_\tau}) = \hat{k}_1(d). \tag{10.16}$$

Since $i(c^m) \ge 3$ for $m > \tau + 1$, we have $H_2(\Lambda, \Lambda^{\kappa_{\tau+1}}) = 0$. Hence the exact sequence

$$H_2(\Lambda^{\kappa_{\tau+1}}, \Lambda^{\kappa_\tau}) \to H_2(\Lambda, \Lambda^{\kappa_\tau}) \to H_2(\Lambda, \Lambda^{\kappa_{\tau+1}})$$

implies that $\dim H_2(\Lambda, \Lambda^{\kappa_\tau}) \le \dim H_2(\Lambda^{\kappa_{\tau+1}}, \Lambda^{\kappa_\tau})$. Using (10.16) we see that

$$\dim H_2(\Lambda, \Lambda^{\kappa_\tau}) \le \hat{k}_1(d). \tag{10.17}$$

The exact sequence

$$H_2(\Lambda, \Lambda^{\kappa_\tau}) \to H_1(\Lambda^{\kappa_\tau}, \Lambda^0) \to H_1(\Lambda, \Lambda^0)$$

and $H_1(\Lambda, \Lambda^0) = \mathbf{Q}$, cf. (7.2), imply

$$\dim H_1(\Lambda^{\kappa_\tau}, \Lambda^0) \le \dim H_2(\Lambda, \Lambda^{\kappa_\tau}) + 1.$$

Using Proposition 10.1 and (10.17) we obtain

$$\tau \le \hat{k}_1(d) + 1. \tag{10.18}$$

Since $\tau = n - 1$, this contradicts (10.7). □

Therefore we have proved that, under Assumption (F), the only prime closed geodesic on $S^2$ cannot be of the class CG-7 with $k_0(c^n) = k_2(c^n) = 0$.

*Remark 10.3* Note that, in Katok's example, for the two closed geodesics $c_1$ and $c_2$ there holds

$$i(c_1) = 1, \quad i(c_1^{m+1}) \ge 3, \quad i(c_2^m) \ge 3, \quad \text{for all } m \ge 1. \tag{10.19}$$

Therefore in (10.9) we have $\tau = 1$ and $H_2(\Lambda, \Lambda^{\kappa_1}) = H_1(\Lambda, \Lambda^{\kappa_1}) = 0$. Thus the long exact homology sequence of the triple $(\Lambda, \Lambda^{\kappa_\tau}, \Lambda^0)$ becomes

$$\begin{array}{ccccccccc} H_2(\Lambda, \Lambda^0) & \longrightarrow & H_2(\Lambda, \Lambda^{\kappa_\tau}) & \longrightarrow & H_1(\Lambda^{\kappa_\tau}, \Lambda^0) & \longrightarrow & H_1(\Lambda, \Lambda^0) & \longrightarrow & H_1(\Lambda, \Lambda^{\kappa_\tau}) \\ \| & & \| & & \| & & \| & & \| \\ 0 & & 0 & & \mathbf{Q} & & \mathbf{Q} & & 0. \end{array} \tag{10.20}$$

This is very different from the case of only one rationally elliptic degenerate saddle closed geodesic.

Now finally we can give

*Proof of Theorem 1.1* Note that, by both Theorem 3.1 and Example 4.1 of [36], it is impossible that the only prime closed geodesic $c$ on $S^2$ in the assumption (F) in the Sect. 7 is of type CG-8 or CG-9 in Sect. 4. Here, for the reader's convenience, we briefly indicate how to exclude these two cases. Note that by $M_1 \geq b_1 = 1$, we must have $i(c) = 1$ in both cases. In Case CG-8, by Theorem 5.3 we get $\alpha_c = 1$ which contradicts that $\alpha_c$ should be irrational. In Case CG-9, we get $i(c^m) = m$ and $\nu(c^m) = 0$ for all $m \geq 1$. Thus similarly to our study in Sect. 9.2, Proposition 3.8 and a direct computation show $M_1 = M_2 = M_3 = 1$. This contradicts the Morse inequality (3.32), since $1 = M_3 - M_2 + M_1 \geq b_3 - b_2 + b_1 = 2$, cf. (7.2) and (7.3).

The preceding Sects. 7–10 show that under the assumption $(F)$ the only prime closed geodesic $c$ cannot be of classes CG-1 to CG-7 either. Therefore the proof of Theorem 1.1 is complete. □

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