Seifert Conjecture in the Even Convex Case

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Abstract

In this paper, we prove that there exist at least n geometrically distinct brake orbits on every C^2 compact convex symmetric hypersurface Σ in $\mathbb{R}^{2n}$ satisfying the reversible condition $N\Sigma = \Sigma$ with $N = \mathrm{diag}(-I_n, I_n)$. As a consequence, we show that if the Hamiltonian function is convex and even, then Seifert conjecture of 1948 on the multiplicity of brake orbits holds for any positive integer n.

1 Introduction

For the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with $\omega_0(x, y) = \langle Jx, y\rangle$, where $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard symplectic matrix and I is the $n \times n$ identity matrix, an involution matrix defined by $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ is clearly antisymplectic, i.e., $NJ = -JN$. The fixed point set of N and $-N$ are the Lagrangian subspaces $L_0 = \{0\} \times \mathbb{R}^n$ and $L_1 = \mathbb{R}^n \times \{0\}$ of $(\mathbb{R}^{2n}, \omega_0)$, respectively.

Suppose $H \in C^2(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R})$ satisfies the reversible condition

$$H(Nx) = H(x) \quad \forall x \in \mathbb{R}^{2n}. \tag{1.1}$$

We consider the following fixed energy problem of a nonlinear Hamiltonian system with Lagrangian boundary conditions:

$$\dot{x}(t) = JH'(x(t)), \tag{1.2}$$

$$H(x(t)) = h, \tag{1.3}$$

$$x(0) \in L_0, x(\tau/2) \in L_0. \tag{1.4}$$

It is clear that a solution (τ, x) of (1.2)–(1.4) is a characteristic chord on the contact submanifold $\Sigma := H^{-1}(h) = \{y \in \mathbb{R}^{2n} \mid H(y) = h\}$ of $(\mathbb{R}^{2n}, \omega_0)$ and satisfies

$$x(-t) = Nx(t), \tag{1.5}$$

$$x(\tau + t) = x(t). \tag{1.6}$$

In this paper this kind of τ-periodic characteristic (τ, x) is called a *brake orbit* on the hypersurface Σ. We denote by $\mathcal{J}_b(\Sigma, H)$ the set of all brake orbits on Σ. Two brake orbits $(\tau_i, x_i) \in \mathcal{J}_b(\Sigma, H)$, $i = 1, 2$, are equivalent if the two brake

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orbits are geometrically the same, i.e., $x_1(\mathbb{R}) = x_2(\mathbb{R})$. We denote by $[(\tau, x)]$ the equivalence class of $(\tau, x) \in \mathcal{J}_b(\Sigma, H)$ in this equivalence relation and by $\tilde{\mathcal{J}}_b(\Sigma, H)$ the set of $[(\tau, x)]$ for all $(\tau, x) \in \mathcal{J}_b(\Sigma, H)$. In fact, $\tilde{\mathcal{J}}_b(\Sigma, H)$ is the set of geometrically distinct brake orbits on Σ, which is independent of the choice of H. So from now on we simply denote it by $\tilde{\mathcal{J}}_b(\Sigma)$ and in the notation $[(\tau, x)]$ we always assume x has minimal period τ. We also denote by $\tilde{\mathcal{J}}(\Sigma)$ the set of all geometrically distinct closed characteristics on Σ. The number of elements in a set S is denoted by $^{\#}S$. It is well-known that $^{\#}\tilde{\mathcal{J}}_b(\Sigma)$ (and also $^{\#}\tilde{\mathcal{J}}(\Sigma)$) is only dependent on Σ; that is to say, for simplicity we take $h = 1$ if H and G are two C^2-functions satisfying (1.1) and $\Sigma_H := H^{-1}(1) = \Sigma_G := G^{-1}(1)$; then $^{\#}\mathcal{J}_b(\Sigma_H) =^{\#} \mathcal{J}_b(\Sigma_G)$.

So we can consider the brake orbit problem in a more general setting. Let Σ be a C^2 compact hypersurface in $\mathbb{R}^{2n}$ bounding a compact set C with nonempty interior. Suppose Σ has nonvanishing Gaussian curvature and satisfies the reversible condition $N(\Sigma - x_0) = \Sigma - x_0 := \{x - x_0 | x \in \Sigma\}$ for some $x_0 \in C$. Without loss of generality, we may assume $x_0 = 0$. We denote the set of all such hypersurfaces in $\mathbb{R}^{2n}$ by $\mathcal{H}_b(2n)$. For $x \in \Sigma$, let $n_\Sigma(x)$ be the unit outward normal vector at $x \in \Sigma$. Note that here by the reversible condition there holds $n_\Sigma(Nx) = Nn_\Sigma(x)$. We consider the dynamics problem of finding $\tau > 0$ and a C^1 smooth curve $x : [0, \tau] \to \mathbb{R}^{2n}$ such that

$$\dot{x}(t) = Jn_\Sigma(x(t)), \quad x(t) \in \Sigma, \tag{1.7}$$

$$x(-t) = Nx(t), \quad x(\tau + t) = x(t), \quad \text{for all } t \in \mathbb{R}. \tag{1.8}$$

A solution (τ, x) of the problem (1.7)-(1.8) determines a brake orbit on Σ.

DEFINITION 1.1. We denote by

$$\mathcal{H}_b^c(2n) = \{\Sigma \in \mathcal{H}_b(2n) \mid \Sigma \text{ is strictly convex}\}, \tag{1.9}$$

$$\mathcal{H}_b^{s,c}(2n) = \{\Sigma \in \mathcal{H}_b^c(2n) \mid -\Sigma = \Sigma\}. \tag{1.10}$$

The main result of this paper is the following:

THEOREM 1.2. *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ there holds*

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq n.$$

Remark 1.3. Theorem 1.2 is a kind of multiplicity result related to the Arnold chord conjecture. The Arnold chord conjecture is an existence result that was proved by K. Mohnke in [24]. Another kind of multiplicity result related to the Arnold chord conjecture was proved in [11].

1.1 Seifert Conjecture

Let us recall the famous conjecture proposed by H. Seifert in his pioneer work [26] concerning the multiplicity of brake orbits in certain Hamiltonian systems in $\mathbb{R}^{2n}$.

As a special case of (1.1), we assume $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ possesses the form

$$H(p,q) = \frac{1}{2}A(q)p \cdot p + V(q), \tag{1.11}$$

where $p, q \in \mathbb{R}^n$, $A(q)$ is a positive definite $n \times n$ for any $q \in \mathbb{R}^n$, and A is C^2, and $V \in C^2(\mathbb{R}^n, \mathbb{R})$ is the potential energy. It is clear that a solution of the Hamiltonian system

$$\dot{x} = JH'(x), \quad x = (p,q), \tag{1.12}$$

$$p(0) = p\left(\frac{\tau}{2}\right) = 0, \tag{1.13}$$

is a brake orbit. Moreover, if h is the total energy of a brake orbit (q, p), i.e., $H(p(t), q(t)) = h$ and $V(q(0)) = V(q(\tau)) = h$, then $q(t) \in \overline{\Omega} \equiv \{q \in \mathbb{R}^n \mid V(q) \le h\}$ for all $t \in \mathbb{R}$.

In [26] of 1948, H. Seifert studied the existence of brake orbit for system (1.12)-(1.13) with the Hamiltonian function H in the form of (1.11) and proved that $\mathcal{J}_b(\Sigma) \neq \emptyset$ provided $V' \neq 0$ on $\partial\Omega$, V is analytic and $\overline{\Omega}$ is bounded and homeomorphic to the unit ball $B_1^n(0)$ in $\mathbb{R}^n$. Then in the same paper he proposed the following conjecture which is still open for $n \ge 2$ now:

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) \ge n \textit{ under the same conditions.}$$

We note that for the Hamiltonian function

$$H(p,q) = \frac{1}{2}|p|^2 + \sum_{j=1}^{n} a_j^2 q_j^2, \qquad q, p \in \mathbb{R}^n,$$

where $a_i/a_j \notin \mathbb{Q}$ for all $i \neq j$ and $q = (q_1, q_2, ..., q_n)$. There are exactly n geometrically distinct brake orbits on the energy hypersurface $\Sigma = H^{-1}(h)$.

1.2 Some Related Results since 1948

As a special case, letting $A(q) = I$ in (1.11), the problem corresponds to the following classical fixed energy problem of the second-order autonomous Hamiltonian system

$$\ddot{q}(t) + V'(q(t)) = 0 \quad \text{for } q(t) \in \Omega, \tag{1.14}$$

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h \quad \forall t \in \mathbb{R}, \tag{1.15}$$

$$\dot{q}(0) = \dot{q}\left(\frac{\tau}{2}\right) = 0, \tag{1.16}$$

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and h is constant such that $\Omega \equiv \{q \in \mathbb{R}^n \mid V(q) < h\}$ is nonempty, bounded, and connected.

A solution (τ, q) of (1.14)–(1.16) is still called a *brake orbit* in $\overline{\Omega}$. Two brake orbits q_1 and $q_2 : \mathbb{R} \to \mathbb{R}^n$ are *geometrically distinct* if $q_1(\mathbb{R}) \neq q_2(\mathbb{R})$. We denote by $\mathcal{O}(\Omega, V)$ and $\widetilde{\mathcal{O}}(\Omega)$ the sets of all brake orbits and geometrically distinct brake orbits in $\overline{\Omega}$, respectively.

Remark 1.4. It is well known that via

$$H(p,q) = \frac{1}{2}|p|^2 + V(q),$$

$x = (p,q)$ and $p = \dot{q}$, the elements in $\mathcal{O}(\Omega, V)$ and the solutions of (1.2)–(1.4) are one-to-one correspondent.

DEFINITION 1.5. For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, a brake orbit (τ, x) on Σ is called *symmetric* if $x(\mathbb{R}) = -x(\mathbb{R})$. Similarly, for a C^2 convex symmetric bounded domain $\Omega \subset \mathbb{R}^n$, a brake orbit $(\tau, q) \in \mathcal{O}(\Omega, V)$ is called *symmetric* if $q(\mathbb{R}) = -q(\mathbb{R})$.

Note that a brake orbit $(\tau, x) \in \mathcal{J}_b(\Sigma, H)$ with minimal period τ is symmetric if $x(t+\tau/2) = -x(t)$ for $t \in \mathbb{R}$, and a brake orbit $(\tau, q) \in \mathcal{O}(\Omega, V)$ with minimal period τ is symmetric if $q(t+\tau/2) = -q(t)$ for $t \in \mathbb{R}$.

Since 1948, many studies have been carried out for the brake orbit problem. In 1978, S. Bolotin proved in [4] the existence of brake orbits in a general setting. K. Hayashi in [12], H. Gluck and W. Ziller in [10], and V. Benci in [2] proved ${}^{\#}\widetilde{\mathcal{O}}(\Omega) \geq 1$ if V is C^1, $\overline{\Omega} = \{V \leq h\}$ is compact, and $V'(q) \neq 0$ for all $q \in \partial\Omega$. P. Rabinowitz in [25] proved that if H satisfies (1.1), $\Sigma \equiv H^{-1}(h)$ is star-shaped, and $x \cdot H'(x) \neq 0$ for all $x \in \Sigma$, then ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) \geq 1$. V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [3]. It has been pointed out in [8] that the problem of finding brake orbits is equivalent to finding orthogonal geodesic chords on a manifold with concave boundary. R. Giambò, F. Giannoni, and P. Piccione in [9] proved the existence of an orthogonal geodesic chord on a Riemannian manifold homeomorphic to a closed disk and with concave boundary.

For multiplicity of the brake problems, A. Weinstein in [30] proved a localized result: *Assume H satisfies* (1.1). *For any h sufficiently close to $H(z_0)$ with z_0 being a nondegenerate local minimum of H, there exist at least n geometrically distinct brake orbits on the energy surface $H^{-1}(h)$.* In [5, 10], under assumptions of Seifert in [26], it was proved that the existence of at least n brake orbits, while a very strong assumption on the energy integral was used to ensure that different minimax critical levels correspond to geometrically distinct brake orbits. A. Szulkin in [27] proved that ${}^{\#}\widetilde{\mathcal{J}}_b(H^{-1}(h)) \geq n$ if H satisfies conditions in [25] of Rabinowitz and the energy hypersurface $H^{-1}(h)$ is $\sqrt{2}$-pinched. E. van Groesen in [28] and A. Ambrosetti, V. Benci, and Y. Long in [1] also proved ${}^{\#}\widetilde{\mathcal{O}}(\Omega) \geq n$ under different pinching conditions. Without a pinching condition, in [21] Y. Long, C. Zhu, and the second author of this paper proved that: *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ with $n \geq 2$, ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) \geq 2$.* The authors of this paper in [17] proved that ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n}{2}\right] + 1$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$. Moreover, it was proved that if all brake orbits on Σ are nondegenerate, then ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) \geq n + \mathfrak{A}(\Sigma)$, where $2\mathfrak{A}(\Sigma)$ is the number of geometrically distinct asymmetric brake orbits on Σ. Recently, in [34] the authors of this paper improved the results of [17] to ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n+1}{2}\right] + 1$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$,

$n \geq 3$. In [33] the authors of this paper proved that ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) \geq \left[\frac{n+1}{2}\right] + 2$ for $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, $n \geq 4$.

1.3 Some Consequences of Theorem 1.2 and Further Arguments

As direct consequences of Theorem 1.2, we have the following two important corollaries:

COROLLARY 1.6. *If $H(p,q)$ defined by (1.11) is even and convex, then the Seifert conjecture holds.*

Remark 1.7. If the function H in Remark 1.3 is convex and even, then V is convex and even, and Ω is convex and central symmetric. Hence Ω is homeomorphic to the unit open ball in $\mathbb{R}^n$.

COROLLARY 1.8. *Suppose $V(0) = 0$, $V(q) \geq 0$, $V(-q) = V(q)$, and $V''(q)$ is positive definite for all $q \in \mathbb{R}^n \setminus \{0\}$. Then for any given $h > 0$ and $\Omega \equiv \{q \in \mathbb{R}^n \mid V(q) < h\}$, there holds*

$$ {}^{\#}\tilde{\mathcal{O}}(\Omega) \geq n. $$

It is interesting to ask the following question: *Are all closed characteristics on any hypersurfaces $\Sigma \in \mathcal{H}_b^{s,c}(2n)$ symmetric brake orbits after suitable time translation provided that ${}^{\#}\tilde{\mathcal{J}}(\Sigma) < +\infty$?* In this direction, we have the following result:

THEOREM 1.9. *For any $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, suppose*

$$ {}^{\#}\tilde{\mathcal{J}}(\Sigma) = n. $$

Then all of the n closed characteristics on Σ are symmetric brake orbits after suitable time translation.

For $n = 2$, it was proved in [13] that ${}^{\#}\tilde{\mathcal{J}}(\Sigma)$ is either 2 or $+\infty$ for any C^2 compact convex hypersurface Σ in $\mathbb{R}^4$. Hence Theorem 1.9 gives a positive answer to the above question in the case $n = 2$. We also note that for the hypersurface

$$ \Sigma = \left\{ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \;\middle|\; x_1^2 + y_1^2 + \frac{x_2^2 + y_2^2}{4} = 1 \right\}, $$

we have ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma) = +\infty$ and ${}^{\#}\tilde{\mathcal{J}}_b^s(\Sigma) = 2$, where we have denoted by $\tilde{\mathcal{J}}_b^s(\Sigma)$ the set of all symmetric brake orbits on Σ. We also note that on the hypersurface $\Sigma = \{x \in \mathbb{R}^{2n} \mid |x| = 1\}$ there are some non-brake-closed characteristics.

The key ingredients in the proof of Theorem 1.2 are some ideas from our previous paper [17] and the following result, which generalizes corresponding results of our previous papers [33,34] completely, where the iteration path γ^2 will be defined in Definition 2.9 below.

THEOREM 1.10. *For $\gamma \in \mathcal{P}_\tau(2n)$, let $P = \gamma(\tau)$. If $i_{L_0}(\gamma) \geq 0$, $i_{L_1}(\gamma) \geq 0$, $i(\gamma) \geq n$, and $\gamma^2(t) = \gamma(t-\tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$, then*

$$ i_{L_1}(\gamma) + S_{P^2}^{+}(1) - \nu_{L_0}(\gamma) \geq 0. \tag{1.17} $$

In this paper, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of positive integers, integers, rational numbers, real numbers, and complex numbers, respectively. We denote by both $\langle \cdot, \cdot \rangle$ and $\cdot$ the standard inner product in $\mathbb{R}^n$ or $\mathbb{R}^{2n}$, and by $(\cdot, \cdot)$ the inner product of corresponding Hilbert space. For any $a \in \mathbb{R}$, we denote by $[a] = \max\{k \in \mathbb{Z} \mid k \leq a\}$.

We prove Theorem 1.2 and Theorem 1.9 in Section 3, and the proof of Theorem 1.10 is given in Sections 4 and 5.

2 Index Theories for Symplectic Paths and the Homotopic Properties of Symplectic Matrices

In this section we make some preparations for the proof of Theorems 1.2 and 1.9. We first briefly introduce the Maslov-type index theory of (i_{L_j}, ν_{L_j}) for $j = 0, 1$ and (i_ω, ν_ω) for $\omega \in \mathbf{U} := \{z \in \mathbb{C} \mid |z| = 1\}$.

Let $\mathcal{L}(\mathbb{R}^{2n})$ denote the set of $2n \times 2n$ real matrices and $\mathcal{L}_s(\mathbb{R}^{2n})$ its subset of symmetric ones. For any $F \in \mathcal{L}_s(\mathbb{R}^{2n})$, we denote by $m^*(F)$ the dimension of maximal positive definite subspace, negative definite subspace, and kernel of any F for $* = +, -, 0$, respectively.

Let

$$J_k = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix} \quad \text{and} \quad N_k = \begin{pmatrix} -I_k & 0 \\ 0 & I_k \end{pmatrix}$$

with I_k being the identity in $\mathbb{R}^k$. If $k = n$ we will omit the subscript k for convenience, i.e., $J_n = J$ and $N_n = N$.

The symplectic group $\mathrm{Sp}(2k)$ for any $k \in \mathbb{N}$ is defined by

$$\mathrm{Sp}(2k) = \{M \in \mathcal{L}(\mathbb{R}^{2k}) \mid M^\top J_k M = J_k\},$$

where $M^\top$ is the transpose of matrix M.

For any $\tau > 0$, the symplectic path in $\mathrm{Sp}(2k)$ starting from the identity I_{2k} is defined by

$$\mathcal{P}_\tau(2k) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2k)) \mid \gamma(0) = I_{2k}\}.$$

The Maslov-type index theory of $(i(\gamma), \nu(\gamma))$ of γ usually plays an important role in the study of periodic solutions of Hamiltonian systems. It was introduced by C. Conley and E. Zehnder in [7] for nondegenerate symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ with $n \geq 2$. Y. Long and E. Zehnder in [23] extended the definition to include $\gamma \in \mathcal{P}_\tau(2)$. Long in [18] and C. Viterbo in [29] further extended the definition for $\gamma \in \mathcal{P}(2n)$. In [19], Long introduced the ω-index, which is an index function $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$ for $\omega \in \mathbf{U}$ (see [20] and [18]).

For any $\omega \in \mathbf{U}$, the following hypersurface in $\mathrm{Sp}(2n)$ is defined by

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \mid \det(M - \omega I_{2n}) = 0\}.$$

For any two continuous paths ξ and η: $[0,\tau] \to \mathrm{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, their joint path is defined by

$$\eta * \xi(t) = \begin{cases} \xi(2t) & \text{if } 0 \le t \le \frac{\tau}{2}, \\ \eta(2t-\tau) & \text{if } \frac{\tau}{2} \le t \le \tau. \end{cases} \tag{2.1}$$

Given any two $(2m_k \times 2m_k)$ matrices of square block form

$$M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

for $k = 1, 2$, as in [20], the $\diamond$-product (or symplectic direct product) of M_1 and M_2 is defined by the following $(2(m_1 + m_2) \times 2(m_1 + m_2))$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

We denote by $M^{\diamond k}$ the k-times self $\diamond$-product of M for any $k \in \mathbb{N}$.

It is easy to see that

$$N_{m_1+m_2}(M_1 \diamond M_2)^{-1} N_{m_1+m_2}(M_1 \diamond M_2) = (N_{m_1} M_1^{-1} N_{m_1} M_1) \diamond (N_{m_2} M_2^{-1} N_{m_2} M_2). \tag{2.2}$$

A special path ξ_n is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\diamond n}, \qquad \forall t \in [0, \tau].$$

DEFINITION 2.1. For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, define

$$\nu_\omega(M) = \dim_{\mathbb{C}} \ker(M - \omega I_{2n}). \tag{2.3}$$

For any $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \tag{2.4}$$

If $\gamma(\tau) \notin \mathrm{Sp}(2n)^{0}_{\omega}$, we define

$$i_\omega(\gamma) = [\mathrm{Sp}(2n)^{0}_{\omega} : \gamma * \xi_n], \tag{2.5}$$

where the right-hand side of (2.5) is the usual homotopy intersection number and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed endpoints. If $\omega = 1$, we will simply write $i(\gamma)$ instead of $i_1(\gamma)$. If $\gamma(\tau) \in \mathrm{Sp}(2n)^{0}_{\omega}$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of γ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf\{ i_\omega(\beta) \mid \beta(\tau) \in U \text{ and } \beta(\tau) \notin \mathrm{Sp}(2n)^{0}_{\omega} \}. \tag{2.6}$$

For any $M \in \mathrm{Sp}(2n)$ we define

$$\Omega(M) = \{P \in \mathrm{Sp}(2n) \mid \sigma(P) \cap \mathbf{U} = \sigma(M) \cap \mathbf{U} \text{ and } \nu_\lambda(P) = \nu_\lambda(M)\ \forall \lambda \in \sigma(M) \cap \mathbf{U}\}, \tag{2.7}$$

where we denote by $\sigma(P)$ the spectrum of P.

We denote by $\Omega^0(M)$ the path-connected component of $\Omega(M)$ containing M, and call it the *homotopy component* of M in $\mathrm{Sp}(2n)$.

DEFINITION 2.2. For any $M_1, M_2 \in \mathrm{Sp}(2n)$, we call $M_1 \approx M_2$ if $M_1 \in \Omega^0(M_2)$.

Remark 2.3. It is easy to check that $\approx$ is an equivalence relation. If $M_1 \approx M_2$, we have $M_1^k \approx M_2^k$ for any $k \in \mathbb{N}$ and $M_1 \diamond M_3 \approx M_2 \diamond M_4$ for $M_3 \approx M_4$. Also we have $M_1 \diamond M_2 \approx M_2 \diamond M_1$ and $PMP^{-1} \approx M$ for any $P, M \in \mathrm{Sp}(2n)$. By theorem 7.8 of [19], $M_1 \diamond M_2 \approx M_1 \diamond M_3$ if and only if $M_2 \approx M_3$.

LEMMA 2.4. *Assume $M_1 \in \mathrm{Sp}(2(k_1+k_2))$ and $M_2 \in \mathrm{Sp}(2k_3)$ have the following block form*:

$$M_1 = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{pmatrix} \quad \textit{and} \quad M_2 = \begin{pmatrix} A_5 & B_5 \\ C_5 & D_5 \end{pmatrix}$$

with $A_1, B_1, C_1, D_1 \in \mathcal{L}(\mathbb{R}^{k_1})$, $A_4, B_4, C_4, D_4 \in \mathcal{L}(\mathbb{R}^{k_1})$, and $A_5, D_5 \in \mathcal{L}(\mathbb{R}^{k_3})$. Let

$$M_3 = \begin{pmatrix} A_1 & 0 & A_2 & B_1 & 0 & B_2 \\ 0 & A_5 & 0 & 0 & B_5 & 0 \\ A_3 & 0 & A_4 & B_3 & 0 & B_4 \\ C_1 & 0 & C_2 & D_1 & 0 & D_2 \\ 0 & C_5 & 0 & 0 & D_5 & 0 \\ C_3 & 0 & C_4 & D_3 & 0 & D_4 \end{pmatrix}.$$

Then

$$M_3 \approx M_1 \diamond M_2. \tag{2.8}$$

PROOF. Let

$$P = \mathrm{diag}\left(\begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & 0 & I_{k_2} \\ 0 & I_{k_3} & 0 \end{pmatrix}, \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & 0 & I_{k_2} \\ 0 & I_{k_3} & 0 \end{pmatrix}\right).$$

It is easy to verify that $P \in \mathrm{Sp}(2(k_1 + k_2 + k_3))$ and $M_3 = P(M_1 \diamond M_2)P^{-1}$. Then (2.8) holds from Remark 2.3 and the proof of Lemma 2.4 is completed. $\square$

The following symplectic matrices were introduced as *basic normal forms* in [20]:

$$D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \lambda = \pm 2,$$

$$N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \qquad \lambda = \pm 1,\ b = \pm 1,\ 0,$$

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in (0,\pi) \cup (\pi, 2\pi),$$

$$N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \qquad \theta \in (0,\pi) \cup (\pi, 2\pi),$$

where

$$b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

with $b_i \in \mathbb{R}$ and $b_2 \neq b_3$.

For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, the *splitting number* of M at ω, defined by

$$S_M^{\pm}(\omega) = \lim_{\epsilon \to 0^+} i_{\omega \exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_\omega(\gamma)$$

for any path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$, possesses the following properties:

LEMMA 2.5 ([19], [20, lemma 9.1.5 and list 9.1.12]). *Splitting numbers $S_M^{\pm}(\omega)$ are well-defined*; *i.e., they are independent of the choice of the path $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$. For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, $S_Q^{\pm}(\omega) = S_M^{\pm}(\omega)$ if $Q \approx M$. Moreover, we have the following*:

(1) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (1,1)$ *for $M = \pm N_1(1,b)$ with $b = 1$ or* 0.
(2) $(S_M^+(\pm 1), S_M^-(\pm 1)) = (0,0)$ *for $M = \pm N_1(1,b)$ with $b = -1$.*
(3) $(S_M^+(e^{\sqrt{-1}\theta}), S_M^-(e^{\sqrt{-1}\theta})) = (0,1)$ *for $M = R(\theta)$ with $\theta \in (0,\pi) \cup (\pi, 2\pi)$.*
(4) $(S_M^+(\omega), S_M^-(\omega)) = (0,0)$ *for $\omega \in \mathbf{U} \setminus \mathbb{R}$ and $M = N_2(\omega, b)$ is* trivial, *i.e., for sufficiently small $\alpha > 0$, $MR((t-1)\alpha)^{\diamond n}$ possesses no eigenvalues on $\mathbf{U}$ for $t \in [0,1)$.*
(5) $(S_M^+(\omega), S_M^-(\omega) = (1,1)$ *for $\omega \in \mathbf{U} \setminus \mathbb{R}$ and $M = N_2(\omega, b)$ is* nontrivial.
(6) $(S_M^+(\omega), S_M^-(\omega) = (0,0)$ *for any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$ with $\sigma(M) \cap \mathbf{U} = \varnothing$.*
(7) $S_{M_1 \diamond M_2}^{\pm}(\omega) = S_{M_1}^{\pm}(\omega) + S_{M_2}^{\pm}(\omega)$ *for any $M_j \in \mathrm{Sp}(2n_j)$ with $j = 1,2$ and $\omega \in \mathbf{U}$.*

We denote by

$$F = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n} \tag{2.9}$$

equipped with the standard inner product $(\cdot,\cdot)$ and define the symplectic structure of F by

$$\{v,w\} = (\mathcal{J}v,w) \quad \forall v,w \in F \text{ where } \mathcal{J} = (-J)\oplus J = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}. \tag{2.10}$$

We denote by $\mathrm{Lag}(F)$ the set of Lagrangian subspaces of F and equip it with the topology as a subspace of the Grassmannian of all $2n$-dimensional subspaces of F.

It is easy to check that, for any $M \in \mathrm{Sp}(2n)$ its graph

$$\mathrm{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \,\middle|\, x \in \mathbb{R}^{2n} \right\}$$

is a Lagrangian subspace of F.

Let

$$V_1 = L_0 \times L_0 = \{0\}\times\mathbb{R}^n\times\{0\}\times\mathbb{R}^n \subset \mathbb{R}^{4n}, \tag{2.11}$$

$$V_2 = L_1 \times L_1 = \mathbb{R}^n\times\{0\}\times\mathbb{R}^n\times\{0\} \subset \mathbb{R}^{4n}. \tag{2.12}$$

By proposition 6.1 of [22] and lemma 2.8 and definition 2.5 of [21], we give the following:

DEFINITION 2.6. For any continuous path $\gamma \in \mathcal{P}_\tau(2n)$, we define the following Maslov-type indices:

$$i_{L_0}(\gamma) = \mu_F^{\mathrm{CLM}}(V_1, \mathrm{Gr}(\gamma), [0,\tau]) - n, \tag{2.13}$$

$$i_{L_1}(\gamma) = \mu_F^{\mathrm{CLM}}(V_2, \mathrm{Gr}(\gamma), [0,\tau]) - n, \tag{2.14}$$

$$\nu_{L_j}(\gamma) = \dim(\gamma(\tau)L_j \cap L_j), \quad j = 0,1, \tag{2.15}$$

where we denote by $i_F^{\mathrm{CLM}}(V,W,[a,b])$ the Maslov index for Lagrangian subspace path pair (V,W) in F on $[a,b]$ defined by Cappell, Lee, and Miller in [6]. For any $M \in \mathrm{Sp}(2n)$ and $j = 0,1$, we also denote by $\nu_{L_j}(M) = \dim(ML_j \cap L_j)$.

The index $i_L(\gamma)$ for any Lagrangian subspace $L \subset \mathbb{R}^{2n}$ and symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ was defined by the first author of this paper in [15] in a different way (see also [14, 21]).

DEFINITION 2.7. Let $\gamma_0, \gamma_1 \in \mathcal{P}_\tau(2n)$ and $j = 0,1$. The paths are called L_j-homotopic, denoted by $\gamma_0 \sim_{L_j} \gamma_1$, if there is a map $\delta : [0,1] \to \mathcal{P}(2n)$ such that $\delta(0) = \gamma_0$ and $\delta(1) = \gamma_1$, and $\nu_{L_j}(\delta(s))$ is constant for $s \in [0,1]$.

LEMMA 2.8 ([15]).

(1) *If* $\gamma_0 \sim_{L_j} \gamma_1$, *then*

$$i_{L_j}(\gamma_0) = i_{L_j}(\gamma_1), \quad \nu_{L_j}(\gamma_0) = \nu_{L_j}(\gamma_1).$$

(2) *If* $\gamma = \gamma_1 \diamond \gamma_2 \in \mathcal{P}(2n)$, *and correspondingly* $L_j = L_j' \oplus L_j''$, *then*

$$i_{L_j}(\gamma) = i_{L_j'}(\gamma_1) + i_{L_j''}(\gamma_2), \quad \nu_{L_j}(\gamma) = \nu_{L_j'}(\gamma_1) + \nu_{L_j''}(\gamma_2).$$

(3) *If $\gamma \in \mathcal{P}(2n)$ is the fundamental solution of*

$$\dot{x}(t) = JB(t)x(t)$$

with symmetric matrix function

$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix}$$

satisfying $b_{22}(t) > 0$ for any $t \in R$, then

$$i_{L_0}(\gamma) = \sum_{0<s<1} \nu_{L_0}(\gamma_s), \quad \gamma_s(t) = \gamma(st).$$

(4) *If $b_{11}(t) > 0$ for any $t \in \mathbb{R}$, then*

$$i_{L_1}(\gamma) = \sum_{0<s<1} \nu_{L_1}(\gamma_s), \quad \gamma_s(t) = \gamma(st).$$

DEFINITION 2.9. For any $\gamma \in \mathcal{P}_\tau$ and $k \in \mathbb{N}$, in this paper the k-time iteration γ^k of $\gamma \in \mathcal{P}_\tau(2n)$ in the brake orbit boundary sense is defined by $\tilde{\gamma}|_{[0,k\tau]}$, where

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t-2j\tau)(N\gamma(\tau)^{-1}N\gamma(\tau))^j, \ t \in [2j\tau, (2j+1)\tau], \ j = 0, 1, \ldots \\ N\gamma(2j\tau + 2\tau - t)N(N\gamma(\tau)^{-1}N\gamma(\tau))^{j+1}, \\ \quad t \in [(2j+1)\tau, (2j+2)\tau], \ j = 0, 1, \ldots \end{cases}$$

3 Proofs of Theorems 1.2 and 1.9

In this section we prove Theorems 1.2 and 1.9.

For $\Sigma \in \mathcal{H}_b^{s,c}(2n)$, let $j_\Sigma : \Sigma \to [0, +\infty)$ be the gauge function of Σ defined by

$$j_\Sigma(0) = 0 \quad \text{and} \quad j_\Sigma(x) = \inf\left\{\lambda > 0 \;\middle|\; \frac{x}{\lambda} \in C\right\} \quad \forall x \in \mathbb{R}^{2n} \setminus \{0\},$$

where C is the domain enclosed by Σ.

Define

$$H_\alpha(x) = (j_\Sigma(x))^\alpha, \ \alpha > 1, \quad H_\Sigma(x) = H_2(x) \quad \forall x \in \mathbb{R}^{2n}. \tag{3.1}$$

Then $H_\Sigma \in C^2(\mathbb{R}^{2n}\setminus\{0\}, \mathbb{R}) \cap C^{1,1}(\mathbb{R}^{2n}, \mathbb{R})$.

We consider the following fixed energy problem:

$$\dot{x}(t) = JH'_\Sigma(x(t)), \tag{3.2}$$

$$H_\Sigma(x(t)) = 1, \tag{3.3}$$

$$x(-t) = Nx(t), \tag{3.4}$$

$$x(\tau + t) = x(t) \quad \forall t \in \mathbb{R}. \tag{3.5}$$

Denote by $\mathcal{J}_b(\Sigma, 2)$ ($\mathcal{J}_b(\Sigma, \alpha)$ for $\alpha = 2$ in (3.1)) the set of all solutions (τ, x) of problem (3.2)–(3.5) and by $\tilde{\mathcal{J}}_b(\Sigma, 2)$ the set of all geometrically distinct solutions of (3.2)–(3.5). By remark 1.2 of [17] or the discussion in [21], elements in $\mathcal{J}_b(\Sigma)$ and $\mathcal{J}_b(\Sigma,2)$ are in one-to-one correspondence. So we have ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma)={}^{\#}\tilde{\mathcal{J}}_b(\Sigma,2)$.

For the readers' convenience, in the following we list some known results that will be used in the proof of Theorem 1.2.

In the following we write $(i_{L_0}(\gamma,k),\nu_{L_0}(\gamma,k))=(i_{L_0}(\gamma^k),\nu_{L_0}(\gamma^k))$ for any symplectic path $\gamma\in\mathcal{P}_\tau(2n)$ and $k\in\mathbb{N}$, where γ^k is defined by Definition 2.9.

LEMMA 3.1 (Theorem 1.5 of [17] and Theorem 4.3 of [22]). *Let* $\gamma_j\in\mathcal{P}_{\tau_j}(2n)$ *for* $j=1,\dots,q$. *Let* $M_j=\gamma_j^2(2\tau_j)=N\gamma_j(\tau_j)^{-1}N\gamma_j(\tau_j)$ *for* $j=1,\dots,q$. *Suppose*

$$\hat{i}_{L_0}(\gamma_j)>0,\quad j=1,\dots,q.$$

Then there exist infinitely many $(R,m_1,m_2,\dots,m_q)\in\mathbb{N}^{q+1}$ *such that*

(i) $\nu_{L_0}(\gamma_j,2m_j\pm1)=\nu_{L_0}(\gamma_j)$,

(ii) $i_{L_0}(\gamma_j,2m_j-1)+\nu_{L_0}(\gamma_j,2m_j-1)=R-(i_{L_1}(\gamma_j)+n+S^+_{M_j}(1)-\nu_{L_0}(\gamma_j))$,

(iii) $i_{L_0}(\gamma_j,2m_j+1)=R+i_{L_0}(\gamma_j)$,

(iv) $\nu(\gamma_j^2,2m_j\pm1)=\nu(\gamma_j^2)$,

(v) $i(\gamma_j^2,2m_j-1)+\nu(\gamma_j^2,2m_j-1)=2R-(i(\gamma_j^2)+2S^+_{M_j}(1)-\nu(\gamma_j^2))$,

(vi) $i(\gamma_j^2,2m_j+1)=2R+i(\gamma_j^2)$,

where we have set $i(\gamma_j^2,n_j)=i(\gamma_j^{2n_j})$, $\nu(\gamma_j^2,n_j)=\nu(\gamma_j^{2n_j})$ *for* $n_j\in\mathbb{N}$.

For any $(\tau,x)\in\mathcal{J}_b(\Sigma,2)$, there is a corresponding path $\gamma_x\in\mathcal{P}_\tau(2n)$. For $m\in\mathbb{N}$, we denote by $i_{L_j}(x,m)=i_{L_j}(\gamma_x^m)$ and $\nu_{L_j}(x,m)=\nu_{L_j}(\gamma_x^m)$ for $j=0,1$. Also we denote $i(x,m)=i(\gamma_x^{2m})$ and $\nu(x,m)=\nu(\gamma_x^{2m})$. We remind the reader that the symplectic path γ_x^m is defined in the interval $[0,\frac{m\tau}{2}]$, and the symplectic path γ_x^{2m} is defined in the interval $[0,m\tau]$. If $m=1$, we denote $i(x)=i(x,1)$ and $\nu(x)=\nu(x,1)$. By lemma 6.3 of [17] we have the following:

LEMMA 3.2. *Suppose* ${}^{\#}\tilde{\mathcal{J}}_b(\Sigma)<+\infty$. *Then there exist an integer* $K\geq0$ *and an injective map* $\phi:\mathbb{N}+K\mapsto\mathcal{J}_b(\Sigma,2)\times\mathbb{N}$ *such that*

(i) *For any* $k\in\mathbb{N}+K$, $[(\tau,x)]\in\mathcal{J}_b(\Sigma,2)$, *and* $m\in\mathbb{N}$ *satisfying* $\phi(k)=([(\tau,x)],m)$, *there holds*

$$i_{L_0}(x,m)\leq k-1\leq i_{L_0}(x,m)+\nu_{L_0}(x,m)-1,$$

where x *has minimal period* τ.

(ii) *For any* $k_j\in\mathbb{N}+K$, $k_1<k_2$, *and* $(\tau_j,x_j)\in\mathcal{J}_b(\Sigma,2)$ *satisfying* $\phi(k_j)=([(\tau_j,x_j)],m_j)$ *with* $j=1,2$ *and* $[(\tau_1,x_1)]=[(\tau_2,x_2)]$, *there holds*

$$m_1<m_2.$$

LEMMA 3.3 (Lemma 7.2 of [17]). *Let* $\gamma\in\mathcal{P}_\tau(2n)$ *be extended to* $[0,+\infty)$ *by* $\gamma(\tau+t)=\gamma(t)\gamma(\tau)$ *for all* $t>0$. *Suppose* $\gamma(\tau)=M=P^{-1}(I_2\diamond\tilde{M})P$ *with*

$\widetilde{M} \in \mathrm{Sp}(2n-2)$ *and* $i(\gamma) \geq n$. *Then we have*

$$i(\gamma, 2) + 2S^+_{M^2}(1) - \nu(\gamma, 2) \geq n + 2.$$

LEMMA 3.4 (Lemma 7.3 of [17]). *For any* $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ *and* $m \in \mathbb{N}$, *there hold*

$$\begin{aligned} i_{L_0}(x, m+1) - i_{L_0}(x, m) &\geq 1, \\ i_{L_0}(x, m+1) + \nu_{L_0}(x, m+1) - 1 &\geq i_{L_0}(x, m+1) \\ &> i_{L_0}(x, m) + \nu_{L_0}(x, m) - 1. \end{aligned}$$

PROOF OF THEOREM 1.2. It is suffices to consider the case ${}^{\#}\widetilde{\mathcal{J}}_b(\Sigma) < +\infty$. Since $-\Sigma = \Sigma$, for $(\tau, x) \in \mathcal{J}_b(\Sigma, 2)$ we have

$$H_\Sigma(x) = H_\Sigma(-x), \quad H'_\Sigma(x) = -H'_\Sigma(-x), \quad H''_\Sigma(x) = H''_\Sigma(-x). \tag{3.6}$$

It follows that $(\tau, -x) \in \mathcal{J}_b(\Sigma, 2)$ and, in view of the definition of γ_x, we obtain that

$$\gamma_x = \gamma_{-x}.$$

Hence

$$\begin{aligned} (i_{L_0}(x, m), \nu_{L_0}(x, m)) &= (i_{L_0}(-x, m), \nu_{L_0}(-x, m)), \\ (i_{L_1}(x, m), \nu_{L_1}(x, m)) &= (i_{L_1}(-x, m), \nu_{L_1}(-x, m)), \end{aligned} \quad \forall m \in \mathbb{N}. \tag{3.7}$$

We can write

$$\begin{aligned} \widetilde{\mathcal{J}}_b(\Sigma, 2) = \{&[(\tau_j, x_j)] \mid j = 1, \ldots, p\} \\ &\cup \{[(\tau_k, x_k)], [(\tau_k, -x_k)] \mid k = p+1, \ldots, p+q\}. \end{aligned} \tag{3.8}$$

with $x_j(\mathbb{R}) = -x_j(\mathbb{R})$ for $j = 1, \ldots, p$ and $x_k(\mathbb{R}) \neq -x_k(\mathbb{R})$ for $k = p+1, \ldots, p+q$. Here we recall that (τ_j, x_j) has minimal period τ_j for $j = 1, \ldots, p+q$ and $x_j(\frac{\tau_j}{2} + t) = -x_j(t)$, $t \in \mathbb{R}$, for $j = 1, \ldots, p$.

In view of Lemma 3.2 there exists an integer $K \geq 0$ and an injective map $\phi : \mathbb{N} + K \to \mathcal{J}_b(\Sigma, 2) \times \mathbb{N}$. By (3.7), (τ_k, x_k) and $(\tau_k, -x_k)$ have the same (i_{L_0}, ν_{L_0})-indices. So by Lemma 3.2, without loss of generality, we can further require that

$$\mathrm{Im}(\phi) \subseteq \{[(\tau_k, x_k)] \mid k = 1, \ldots, p+q\} \times \mathbb{N}. \tag{3.9}$$

By the strict convexity of H_Σ and (6.19) of [17]), we have

$$\hat{i}_{L_0}(x_k) > 0, \quad k = 1, \ldots, p+q.$$

Applying Lemma 3.1 to symplectic paths

$$\gamma_1, \ldots, \gamma_{p+q}, \gamma_{p+q+1}, \ldots, \gamma_{p+2q}$$

associated with $(\tau_1, x_1), \ldots, (\tau_{p+q}, x_{p+q}), (2\tau_{p+1}, x^2_{p+1}), \ldots, (2\tau_{p+q}, x^2_{p+q})$, respectively, there exists a vector $(R, m_1, \ldots, m_{p+2q}) \in \mathbb{N}^{p+2q+1}$ such that $R > K + n$ and

$$i_{L_0}(x_k, 2m_k + 1) = R + i_{L_0}(x_k), \tag{3.10}$$

$$(3.11)\quad i_{L_0}(x_k, 2m_k - 1) + \nu_{L_0}(x_k, 2m_k - 1) = R - \big(i_{L_1}(x_k) + n + S^+_{M_k}(1) - \nu_{L_0}(x_k)\big)$$

for $k = 1, \dots, p+q$, $M_k = \gamma_k^2(\tau_k)$, and

$$(3.12)\quad i_{L_0}(x_k, 4m_k + 2) = R + i_{L_0}(x_k, 2),$$

$$(3.13)\quad i_{L_0}(x_k, 4m_k - 2) + \nu_{L_0}(x_k, 4m_k - 2) = R - \big(i_{L_1}(x_k, 2) + n + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2)\big)$$

for $k = p+q+1, \dots, p+2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

By Lemma 3.1, we also have

$$(3.14)\quad i(x_k, 2m_k + 1) = 2R + i(x_k),$$

$$(3.15)\quad i(x_k, 2m_k - 1) + \nu(x_k, 2m_k - 1) = 2R - (i(x_k) + 2S^+_{M_k}(1) - \nu(x_k)),$$

for $k = 1, \dots, p+q$, $M_k = \gamma_k^2(\tau_k)$, and

$$(3.16)\quad i(x_k, 4m_k + 2) = 2R + i(x_k, 2),$$

$$(3.17)\quad i(x_k, 4m_k - 2) + \nu(x_k, 4m_k - 2) = 2R - \big(i(x_k, 2) + 2S^+_{M_k}(1) - \nu(x_k, 2)\big),$$

for $k = p+q+1, \dots, p+2q$ and $M_k = \gamma_k^4(2\tau_k) = \gamma_k^2(\tau_k)^2$.

From (3.9), we can set

$$\phi(R - (s-1)) = ([(\tau_{k(s)}, x_{k(s)})], m(s)) \quad \forall s \in S := \{1, \dots, n\},$$

where $k(s) \in \{1, \dots, p+q\}$ and $m(s) \in \mathbb{N}$.

We continue our proof to study the symmetric and asymmetric orbits separately. Let

$$S_1 = \{s \in S \mid k(s) \le p\}, \quad S_2 = S \setminus S_1.$$

We shall prove that ${}^\# S_1 \le p$ and ${}^\# S_2 \le 2q$. These estimates together with the definitions of S_1 and S_2 yield Theorem 1.2.

Claim 3.5. ${}^\# S_1 \le p$.

PROOF. By the definition of S_1, we have that $([(\tau_{k(s)}, x_{k(s)})], m(s))$ is symmetric when $k(s) \le p$. We further prove that $m(s) = 2m_{k(s)}$ for $s \in S_1$.

In fact, by the definition of ϕ and Lemma 3.2, for all $s = 1, \dots, n$ we have

$$\begin{aligned} i_{L_0}(x_{k(s)}, m(s)) &\le (R - (s-1)) - 1 = R - s \\ &\le i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \end{aligned}$$

By the strict convexity of H_Σ and Lemma 2.8, we have $i_{L_0}(x_{k(s)}) \ge 0$, so that

$$(3.18)\quad \begin{aligned} i_{L_0}(x_{k(s)}, m(s)) \le R - s < R \le R + i_{L_0}(x_{k(s)}) \\ = i_{L_0}(x_{k(s)}, 2m_{k(s)} + 1), \end{aligned}$$

for every $s = 1, \dots, n$, where we have used (3.10) in the last equality. Note that the proofs of (3.18) and (3.18) do not depend on the condition $s \in S_1$.

It is easy to see that γ_{x_k} satisfies the conditions of Theorem 1.10 with $\tau = \tau_k/2$. Note that by definition $i_{L_1}(x_k) = i_{L_1}(\gamma_{x_k})$ and $\nu_{L_0}(x_k) = \nu_{L_0}(\gamma_{x_k})$. So by Theorem 1.10 we have

$$i_{L_1}(x_k) + S^+_{M_k}(1) - \nu_{L_0}(x_k) \geq 0 \quad \forall k = 1, \dots, p. \tag{3.19}$$

Hence by (3.18) and (3.19), if $k(s) \leq p$, it follows that

$$\begin{aligned} & i_{L_0}(r x_{k(s)}, 2m_{k(s)} - 1) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 1) - 1 \\ & \quad = R - (i_{L_1}(x_{k(s)}) + n + S^+_{M_{k(s)}}(1) - \nu_{L_0}(x_{k(s)})) - 1 \\ & \quad \leq R - \frac{1-n}{2} - 1 - n \\ & \quad < R - s \\ & \quad \leq i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \end{aligned} \tag{3.20}$$

Thus by (3.18), (3.20), and Lemma 3.4 we obtain

$$2m_{k(s)} - 1 < m(s) < 2m_{k(s)} + 1.$$

Hence

$$m(s) = 2m_{k(s)} \quad \text{and} \quad \phi(R - s + 1) = ([(\tau_{k(s)}, x_{k(s)})], 2m_{k(s)}) \quad \forall s \in S_1.$$

Then the injectivity of the map ϕ induces an injective map

$$\phi_1 : S_1 \to \{1, \dots, p\}, \quad s \mapsto k(s).$$

Therefore, ${}^{\#}S_1 \leq p$ and Claim 3.5 is proved. $\square$

Claim 3.6. ${}^{\#}S_2 \leq 2q$.

PROOF. By the formulas (3.14)–(3.17), and (59) of [16] (also [20, claim 4, p. 352]), we have

$$m_k = 2m_{k+q} \quad \text{for } k = p+1, p+2, \dots, p+q. \tag{3.21}$$

By Theorem 1.10 there holds

$$i_{L_1}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2) \geq 0, \quad p + 1 \leq k \leq p + q. \tag{3.22}$$

By (3.13), (3.18), (3.21) and (3.22), for $p + 1 \leq k(s) \leq p + q$ we have

$$\begin{aligned} & i_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) + \nu_{L_0}(x_{k(s)}, 2m_{k(s)} - 2) - 1 \\ & \quad = i_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) + \nu_{L_0}(x_{k(s)}, 4m_{k(s)+q} - 2) - 1 \\ & \quad = R - (i_{L_1}(x_{k(s)}, 2) + n + S^+_{M_{k(s)}}(1) - \nu_{L_0}(x_{k(s)}, 2)) - 1 \\ & \quad = R - (i_{L_1}(x_k, 2) + S^+_{M_k}(1) - \nu_{L_0}(x_k, 2)) - 1 - n \\ & \quad \leq R - 1 - n < \end{aligned}$$

$$< R - s$$

$$\leq i_{L_0}(x_{k(s)}, m(s)) + \nu_{L_0}(x_{k(s)}, m(s)) - 1. \tag{3.23}$$

Thus (3.18), (3.23) and Lemma 3.4 imply

$$2m_{k(s)} - 2 < m(s) < 2m_{k(s)} + 1, \quad p < k(s) \leq p + q.$$

So

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\} \quad \text{for } p < k(s) \leq p + q.$$

In particular, this yields that for any s_0 and $s \in S_2$, if $k(s) = k(s_0)$, then

$$m(s) \in \{2m_{k(s)} - 1, 2m_{k(s)}\} = \{2m_{k(s_0)} - 1, 2m_{k(s_0)}\}.$$

Then, in view of the injectivity of the map ϕ from Lemma 3.2, we have

$$^{\#}\{s \in S_2 \mid k(s) = k(s_0)\} \leq 2.$$

This proves Claim 3.6. □

By Claim3.5 and Claim 3.6, we obtain

$$^{\#}\tilde{\mathcal{J}}_b(\Sigma) =^{\#} \tilde{\mathcal{J}}_b(\Sigma, 2) = p + 2q \geq^{\#} S_1 +^{\#} S_2 = n.$$

The proof of Theorem 1.2 is completed. □

PROOF OF THEOREM 1.9. We call a closed characteristic x on Σ a *dual brake orbit* on Σ if $x(-t) = -Nx(t)$. Then by the similar proof of lemma 3.1 of [31], a closed characteristic x on Σ can become a dual brake orbit after suitable time translation if and only if $x(\mathbb{R}) = -Nx(\mathbb{R})$. So by lemma 3.1 of [31] again, if a closed characteristic x on Σ can both become brake orbits and dual brake orbits after suitable translation, then $x(\mathbb{R}) = Nx(\mathbb{R}) = -Nx(\mathbb{R})$. Thus $x(\mathbb{R}) = -x(\mathbb{R})$.

Since we also have $-N\Sigma = \Sigma$, $(-N)^2 = I_{2n}$, and $(-N)J = -J(-N)$, dually by the same proof of Theorem 1.2 (with the estimate (5.3) in Theorem 5.3 below), there are at least n geometrically distinct dual brake orbits on Σ.

If there are exactly n closed characteristics on Σ, then Theorem 1.2 implies that all of them are brake orbits on Σ after suitable time translation. By the same argument all the n closed characteristics must be dual brake orbits on Σ. Then by the argument in the first paragraph of the proof of this theorem, all these n closed characteristics on Σ must be symmetric. Hence all of them are symmetric brake orbits after suitable time translation. The proof of Theorem 1.9 is completed. □

4 (L_0, L_1)-Concavity and (ε, L_0, L_1)-Signature of Symplectic Matrix

DEFINITION 4.1. For any $P \in \mathrm{Sp}(2n)$ and $\varepsilon \in \mathbb{R}$, we define the (ε, L_0, L_1)-*symmetrization of P* by

$$M_\varepsilon(P) = P^\top \begin{pmatrix} \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} P + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.$$

The (ε, L_0, L_1)-*signature of* P is defined by the signature of $M_\varepsilon(P)$. The (L_0, L_1)-*concavity* and $(L_0, L_1)^*$-*concavity of a symplectic path* γ is defined by

$$\text{concav}_{(L_0,L_1)}(\gamma) = i_{L_0}(\gamma) - i_{L_1}(\gamma),$$
$$\text{concav}^*_{(L_0,L_1)}(\gamma) = (i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)),$$

respectively.

In [15] it was proved that (L_0, L_1)-concavity is only dependent on the end matrix $\gamma(\tau)$ of γ, and in [32] it was proved that the (L_0, L_1)-concavity of a symplectic path γ is half of the (ε, L_0, L_1)-signature of $\gamma(\tau)$; i.e., we have the following result:

THEOREM 4.2 ([32]). *For* $\gamma \in \mathcal{P}_\tau(2k)$ *with* $\tau > 0$, *we have*

$$\text{concav}_{(L_0,L_1)}(\gamma) = \frac{1}{2}\,\text{sgn}\, M_\varepsilon(\gamma(\tau)),$$

where $0 < \varepsilon \ll 1$, *and we have denoted by* sgn A *the signature of* A *for any symmetric matrix* A. *We also have*

$$\text{concav}^*_{(L_0,L_1)}(\gamma) = \frac{1}{2}\,\text{sgn}\, M_\varepsilon(\gamma(\tau)), \quad 0 < -\varepsilon \ll 1.$$

Remark 4.3 (Remark 2.1 of [32]). For any $2n_j \times 2n_j$ symplectic matrix P_j with $j = 1, 2$ and $n_j \in \mathbb{N}$, we have

$$M_\varepsilon(P_1 \diamond P_2) = M_\varepsilon(P_1) \diamond M_\varepsilon(P_2),$$
$$\text{sgn}\, M_\varepsilon(P_1 \diamond P_2) = \text{sgn}\, M_\varepsilon(P_1) + \text{sgn}\, M_\varepsilon(P_2),$$

where $\varepsilon \in \mathbb{R}$.

In the rest of this section, we further develop some basic properties of the (ε, L_0, L_1)-signature and study the normal forms of L_0-degenerate symplectic matrices.

LEMMA 4.4 (Lemma 2.3 of [34]). *Let* $k \in \mathbb{N}$ *and let*

$$P = \begin{pmatrix} I_k & 0 \\ C & I_k \end{pmatrix}$$

be any symplectic matrix. Then $P \approx I_2^{\diamond p} \diamond N_1(1,1)^{\diamond q} \diamond N_1(1,-1)^{\diamond r}$ *with* $p = m^0(C)$, $q = m^-(C)$, *and* $r = m^+(C)$.

DEFINITION 4.5. We call two symplectic matrices M_1 and M_2 (L_0, L_1)-*homotopic equivalent* in $\text{Sp}(2k)$, and denote the relationship by $M_1 \sim M_2$, if there are $P_j \in \text{Sp}(2k)$ of the form $P_j = \text{diag}(Q_j, (Q_j^{\mathsf{T}})^{-1})$, where Q_j is a $k \times k$ invertible real matrix with $\det(Q_j) > 0$ for $j = 1, 2$ such that

$$M_1 = P_1 M_2 P_2.$$

Remark 4.6. Let

$$M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathrm{Sp}(2k_i), \quad i = 0, 1, 2,$$

$k_1 = k_2$, and $M_1 \sim M_2$; then $A_1^\top C_1$ and $B_1^\top D_1$ are congruent to $A_2^\top C_2$ and $B_2^\top D_2$, respectively. So $m^*(A_1^\top C_1) = m^*(A_2^\top C_2)$ and $m^*(B_1^\top D_1) = m^*(B_2^\top D_2)$ for $* = \pm, 0$. Furthermore, if $M_0 = M_1 \diamond M_2$ (here $k_1 = k_2$ is not necessary), then

$$\begin{aligned} m^*(A_0^\top C_0) &= m^*(A_1^\top C_1) + m^*(A_2^\top C_2), \\ m^*(B_0^\top D_0) &= m^*(B_1^\top D_1) + m^*(B_2^\top D_2), \end{aligned} \tag{4.1}$$

and so $m^*(A^\top C)$ and $m^*(B^\top D)$ are (L_0, L_1)-homotopic invariant. The following formula will be used frequently:

$$N_k M_1^{-1} N_k M_1 = I_{2k} + 2 \begin{pmatrix} B_1^\top C_1 & B_1^\top D_1 \\ A_1^\top C_1 & C_1^\top B_1 \end{pmatrix}. \tag{4.2}$$

It is clear that $\sim$ is an equivalence relation and we have the following lemma:

LEMMA 4.7 (Lemma 2.4 of [34]). *For $M_1, M_2 \in \mathrm{Sp}(2k)$, if $M_1 \sim M_2$, then*

$$\begin{gathered} \operatorname{sgn} M_\varepsilon(M_1) = \operatorname{sgn} M_\varepsilon(M_2), \quad 0 \le |\varepsilon| \ll 1, \\ N_k M_1^{-1} N_k M_1 \approx N_k M_2^{-1} N_k M_2. \end{gathered}$$

By results in [32–34], we have the following Lemmas 4.8–4.10, which will be used frequently in Section 4.

LEMMA 4.8 (Lemma 2.5 of [34]). *Assume*

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2k),$$

where A, B, C, and D are all $k \times k$ matrices.

(i) *Let $q = \max\{m^+(A^\top C), m^+(B^\top D)\}$; we have*

$$\begin{aligned} \frac{1}{2} \operatorname{sgn} M_\varepsilon(P) &\le k - q - \nu_{L_1}(P), \quad 0 < -\varepsilon \ll 1, \\ \frac{1}{2} \operatorname{sgn} M_\varepsilon(P) &\le k - q - \nu_{L_0}(P), \quad 0 < \varepsilon \ll 1. \end{aligned}$$

(ii) *If both B and C are invertible, then*

$$\operatorname{sgn} M_\varepsilon(P) = \operatorname{sgn} M_0(P), \quad 0 \le |\varepsilon| \ll 1.$$

LEMMA 4.9 ([32]). *For $\gamma \in \mathcal{P}_\tau(2)$, $b > 0$, and $\varepsilon > 0$ small enough we have*

$$\begin{aligned} &\operatorname{sgn} M_{\pm\varepsilon}(R(\theta)) = 0 \quad \textit{for } \theta \in \mathbb{R}, \\ &\operatorname{sgn} M_{\pm\varepsilon}(P) = 0 \qquad \textit{if } P = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \textit{ with } a \in \mathbb{R} \setminus \{0\}, \end{aligned}$$

$$\operatorname{sgn} M_\varepsilon(P) = 0 \qquad \textit{if } P = \pm\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \textit{ or } \pm\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix},$$
$$\operatorname{sgn} M_\varepsilon(P) = 2 \qquad \textit{if } P = \pm\begin{pmatrix}1 & -b\\ 0 & 1\end{pmatrix},$$
$$\operatorname{sgn} M_\varepsilon(P) = -2 \qquad \textit{if } P = \pm\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}.$$

LEMMA 4.10 (Lemma 2.9 of [33]). *Let $2k \times 2k$ symmetric real matrix E have the block form*

$$E = \begin{pmatrix}0 & E_1\\ E_1^{\mathsf{T}} & E_2\end{pmatrix}.$$

Then

$$m^{\pm}(E) \ge \operatorname{rank} E_1. \tag{4.3}$$

Lemma 4.13 and Lemma 4.14 are key technical results of this paper. The next lemma is used in the proof of Lemma 4.13.

LEMMA 4.11. *Let A_1 and A_3 be $k \times k$ real matrices. Assume that both A_1 and A_1A_3 are symmetric and $\sigma(A_3) \subset (-\infty, 0)$. Then*

$$\operatorname{sgn} A_1 + \operatorname{sgn}(A_1A_3) = 0. \tag{4.4}$$

PROOF. It is clear that A_3 is invertible. We prove Lemma 4.11 in the following two steps.

Step 1. We assume that A_1 is invertible and proceed by induction on $k \in \mathbb{N}$.

If $k = 1$, then $A_1, A_3 \in \mathbb{R}$ and (4.4) obviously holds. Now assume (4.4) holds for $1 \le k \le l$. If we can prove (4.4) for $k = l + 1$, then by mathematical induction (4.4) holds for any $k \in \mathbb{N}$ and Lemma 4.11 is proved in the case A_1 is invertible.

In view of the real Jordan canonical form decomposition of A_3, we only need to prove (4.4) for $k = l + 1$ in the following two cases.

Case 1. There is an invertible $(l + 1) \times (l + 1)$ real matrix such that $Q^{-1}A_3Q$ is the $(l + 1)$-order Jordan form

$$\begin{pmatrix}\lambda & 1 & 0 & \cdots & \cdots & 0\\ 0 & \lambda & 1 & \ddots & \mathbf{0} & \vdots\\ 0 & 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & 0\\ \vdots & \mathbf{0} & \ddots & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & 0 & \lambda\end{pmatrix} := \widetilde{A}_3$$

with $\lambda < 0$.

Denoting by $\widetilde{A}_1 = Q^{\mathsf{T}}A_1Q$, we have

$$\widetilde{A}_1\widetilde{A}_3 = Q^{\mathsf{T}}A_1Q\,Q^{-1}A_3Q = Q^{\mathsf{T}}A_1A_3Q.$$

Hence both matrices $\widetilde{A}_1$ and $\widetilde{A}_1\widetilde{A}_3$ are symmetric and

$$\operatorname{sgn} A_1 + \operatorname{sgn}(A_1A_3) = \operatorname{sgn}\widetilde{A}_1 + \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3). \tag{4.5}$$

Since $\widetilde{A}_1 = (a_{i,j})_{1\le i,j\le l+1}$ and $\widetilde{A}_1\widetilde{A}_3 = (c_{i,j})_{1\le i,j\le l+1}$ are symmetric, $a_{i,j} = a_{j.i}$ and $c_{i,j} = c_{j,i}$ for $1 \le i, j \le l+1$.

Claim 4.12. $a_{i,j} = 0$ for $i + j \le l + 1$ and $a_{i,j} = a_{l+1,1}$ for $i + j = l + 2$ with $1 \le i, j \le l + 1$.

PROOF. For $2 \le j \le l+1$, since $c_{1,j} = c_{j,1}$,

$$\lambda a_{1,j} + a_{1,j-1} = \lambda a_{j,1} = \lambda a_{1,j}.$$

Thus

$$a_{1,j-1} = 0, \quad 2 \le j \le l+1. \tag{4.6}$$

For $2 \le i, j \le l+1$, since $c_{i,j} = c_{j,i}$ we have

$$\lambda a_{i,j} + a_{i,j-1} = \lambda a_{j,i} + a_{j,i-1} = \lambda a_{i,j} + a_{i-1,j}.$$

So

$$a_{i,j-1} = a_{i-1,j}, \quad 2 \le i, j \le l+1. \tag{4.7}$$

By (4.6) and (4.7) we have

$$\begin{gathered} a_{i,j} = a_{i-1,j+1} = \cdots = a_{2,i+j-2} = a_{1,i+j-1} = 0, \\ 1 \le i, j \quad \text{and} \quad i + j \le l+1, \end{gathered} \tag{4.8}$$

$$a_{l+1,1} = a_{l,2} = a_{l-1,3} = \cdots = a_{2,l} = a_{1,l+1}. \tag{4.9}$$

Hence, by (4.8) and (4.9), Claim 4.12 is proved. $\square$

By Claim 4.12, let $a = a_{1,l+1}$; then

$$\begin{gathered} \widetilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & * \\ 0 & 0 & 0 & 0 & \cdot & * & * \\ 0 & 0 & 0 & \cdot & * & * & * \\ 0 & 0 & \cdot & * & * & * & * \\ 0 & a & * & * & * & * & * \\ a & * & * & * & * & * & * \end{pmatrix}, \\ \widetilde{A}_1\widetilde{A}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \lambda a \\ 0 & 0 & 0 & 0 & 0 & \lambda a & * \\ 0 & 0 & 0 & 0 & \cdot & * & * \\ 0 & 0 & 0 & \cdot & * & * & * \\ 0 & 0 & \cdot & * & * & * & * \\ 0 & \lambda a & * & * & * & * & * \\ \lambda a & * & * & * & * & * & * \end{pmatrix}. \end{gathered} \tag{4.10}$$

It is easy to see that $\widetilde{A}_1\widetilde{A}_3$ is congruent to $\lambda\widetilde{A}_1$. Since $\lambda < 0$,

$$\begin{aligned} \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) &= \operatorname{sgn}(\lambda\widetilde{A}_1) = -\operatorname{sgn}(\widetilde{A}_1), \\ \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) &+ \operatorname{sgn}\widetilde{A}_1 = 0. \end{aligned} \tag{4.11}$$

(4.5) and (4.11) imply (4.4). Hence Step 1 is proved in Case 1.

Case 2. There exists an invertible $(l+1)\times(l+1)$ real matrix Q such that $Q^{-1}A_3Q = \operatorname{diag}(A_4, A_5)$, where A_4 is a $k_1 \times k_1$ real matrix with $\sigma(A_4) \subset (-\infty, 0)$ and A_5 is a k_2-order Jordan form

$$A_5 = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \mathbf{0} & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

with $\lambda < 0$, $1 \le k_1, k_2 \le l$, and $k_1 + k_2 = l + 1$.

We still denote $\widetilde{A}_1 = Q^\top A_1 Q$; then

$$\widetilde{A}_1\widetilde{A}_3 = Q^\top A_1 Q\, Q^{-1} A_3 Q = Q^\top A_1 A_3 Q.$$

So both $\widetilde{A}_1$ and $\widetilde{A}_1\widetilde{A}_3$ are symmetric and

$$\operatorname{sgn} A_1 + \operatorname{sgn}(A_1A_3) = \operatorname{sgn}\widetilde{A}_1 + \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3). \tag{4.12}$$

Correspondingly, we can write $\widetilde{A}_1$ in the block form decomposition

$$\widetilde{A}_1 = \begin{pmatrix} E_1 & E_2 \\ E_2^\top & E_4 \end{pmatrix},$$

where E_1 is a $k_1 \times k_1$ real symmetric matrix and E_4 is a $k_2 \times k_2$ real symmetric matrix. Then

$$\widetilde{A}_1\widetilde{A}_3 = \begin{pmatrix} E_1A_4 & E_2A_5 \\ E_2^\top A_4 & E_4A_5 \end{pmatrix}$$

is symmetric.

SUBCASE 1. E_4 is invertible.

In this case we have

$$\begin{pmatrix} I_{k_1} & -E_2E_4^{-1} \\ 0 & I_{k_2} \end{pmatrix}\begin{pmatrix} E_1 & E_2 \\ E_2^\top & E_4 \end{pmatrix}\begin{pmatrix} I_{k_1} & 0 \\ -E_4^{-1}E_2^\top & I_{k_2} \end{pmatrix} = \begin{pmatrix} E_1 - E_2E_4^{-1}E_2^\top & 0 \\ 0 & E_4 \end{pmatrix} \tag{4.13}$$

and

$$\begin{pmatrix} I_{k_1} & -E_2E_4^{-1} \\ 0 & I_{k_2} \end{pmatrix} \begin{pmatrix} E_1A_4 & E_2A_5 \\ E_2^{\mathsf{T}}A_4 & E_4A_5 \end{pmatrix} \begin{pmatrix} I_{k_1} & 0 \\ -E_4^{-1}E_2^{\mathsf{T}} & I_{k_2} \end{pmatrix}$$
$$(4.14) \qquad = \begin{pmatrix} E_1A_4 - E_2E_4^{-1}E_2^{\mathsf{T}}A_4 & 0 \\ 0 & E_4A_5 \end{pmatrix}$$
$$= \begin{pmatrix} (E_1 - E_2E_4^{-1}E_2^{\mathsf{T}})A_4 & 0 \\ 0 & E_4A_5 \end{pmatrix}.$$

Since the matrices $\widetilde{A}_1$ and $\widetilde{A}_1\widetilde{A}_3$ are symmetric and invertible, by (4.13) and (4.14), both $E_1 - E_2E_4^{-1}E_2^{\mathsf{T}}$ and $(E_1 - E_2E_4^{-1}E_2^{\mathsf{T}})A_4$ are symmetric and invertible. Hence from $1 \le k_1 \le l$, $\sigma(A_4) \subset (-\infty, 0)$, and our induction hypothesis we obtain

$$(4.15) \qquad \operatorname{sgn}\big((E_1 - E_2E_4^{-1}E_2^{\mathsf{T}})A_4\big) + \operatorname{sgn}\big(E_1 - E_2E_4^{-1}E_2^{\mathsf{T}}\big) = 0.$$

By (4.14), E_4A_5 is symmetric. Since E_4 is symmetric and invertible, $\sigma(A_5) \subset (-\infty, 0)$ and $1 \le k_2 \le l$, by our induction hypothesis we have

$$(4.16) \qquad \operatorname{sgn}(E_4A_5) + \operatorname{sgn} E_4 = 0.$$

From (4.13) we obtain

$$(4.17) \qquad \operatorname{sgn} \widetilde{A}_1 = \operatorname{sgn}\big(E_1 - E_2E_4^{-1}E_2^{\mathsf{T}}\big) + \operatorname{sgn} E_4.$$

By (4.14) there holds

$$(4.18) \qquad \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) = \operatorname{sgn}\big((E_1 - E_2E_4^{-1}E_2^{\mathsf{T}})A_4\big) + \operatorname{sgn}(E_4A_5).$$

Then by (4.15)–(4.18) we have

$$(4.19) \qquad \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) + \operatorname{sgn} \widetilde{A}_1 = 0.$$

Therefore, (4.12) and (4.19) imply (4.4).

SUBCASE 2. E_4 is not invertible.

In this case we define k_2-order real invertible matrix

$$E_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that E_0A_5 is symmetric and $E_4 + \varepsilon E_0$ is invertible for $0 < \varepsilon \ll 1$. Define

$$A_\varepsilon = \begin{pmatrix} E_1 & E_2 \\ E_2^{\mathsf{T}} & E_4 + \varepsilon E_0 \end{pmatrix}.$$

Since $\widetilde{A}_1$ and $\widetilde{A}_1\widetilde{A}_3$ are invertible, we have that both A_ε and $A_\varepsilon\widetilde{A}_3$ are symmetric and invertible. Thus

$$\operatorname{sgn}\widetilde{A}_1 = \operatorname{sgn} A_\varepsilon, \quad \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) = \operatorname{sgn}(A_\varepsilon\widetilde{A}_3) \quad \text{for } 0 < \varepsilon \ll 1. \tag{4.20}$$

By the proof of Subcase 1, we have

$$\operatorname{sgn}(A_\varepsilon\widetilde{A}_3) + \operatorname{sgn} A_\varepsilon = 0. \tag{4.21}$$

So from (4.20) we obtain

$$\operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) + \operatorname{sgn}\widetilde{A}_1 = 0. \tag{4.22}$$

Then (4.4) holds from (4.22).

So in Case 2 (4.4) holds for $k = l + 1$. Hence in the case A_1 is invertible, Lemma 4.11 holds and Step 1 is finished.

Step 2. We assume that A_1 is not invertible.

If $A_1 = 0$, (4.4) obviously holds.

If $1 \le \operatorname{rank} A_1 = m \le k - 1$, there is a real orthogonal matrix G such that

$$G^\top A_1 G = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{A}_1 \end{pmatrix}, \tag{4.23}$$

where $\widehat{A}_1$ is an m^{th}-order invertible real symmetric matrix. Correspondingly, we write

$$G^{-1} A_3 G = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix},$$

where F_1 is a $(k-m)\times(k-m)$ real matrix and F_4 is a $m \times m$ real matrix.

Since A_1A_3 is symmetric, from

$$G^\top A_1 A_3 G = G^\top A_1 G G^{-1} A_3 G = \begin{pmatrix} 0 & 0 \\ \widehat{A}_1 F_3 & \widehat{A}_1 F_4 \end{pmatrix},$$

we get $\widehat{A}_1 F_3 = 0$. Hence $F_3 = 0$ by the invertibility of $\widehat{A}_1$. Therefore can write

$$G^{-1} A_3 G = \begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix}. \tag{4.24}$$

Hence

$$G^\top A_1 A_3 G^\top = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{A}_1 F_4 \end{pmatrix}, \tag{4.25}$$

where $\widehat{A}_1 F_4$ is symmetric. Also, by (4.24) the matrix F_4 is invertible and $\sigma(F_4) \subset (-\infty, 0)$. Thus by the proof of Step 1, there holds

$$\operatorname{sgn}(\widehat{A}_1 F_4) + \operatorname{sgn}\widehat{A}_1 = 0. \tag{4.26}$$

Identities (4.23) and (4.25) give

$$\operatorname{sgn}(A_1A_3) + \operatorname{sgn} A_1 = \operatorname{sgn}(\widehat{A}_1 F_4) + \operatorname{sgn}\widehat{A}_1. \tag{4.27}$$

Then (4.26) and (4.27) give (4.4). Hence Step 2 is proved.

By Step 1 and Step 2 Lemma 4.11 holds. □

We recall that the elliptic height $e(P)$ of P is the total algebraic multiplicity of all eigenvalues of P on $\mathbf{U}$ for any $P \in \mathrm{Sp}(2n)$ (cf. [20, def. 1.8.1]).

LEMMA 4.13. *Let*

$$R = \begin{pmatrix} A_1 & I_k \\ A_3 & A_2 \end{pmatrix} \in \mathrm{Sp}(2k)$$

with A_3 being invertible. If $e(N_k R^{-1} N_k R) = 2m$, where $0 \le m \le k$, then

$$m - k \le \frac{1}{2} \operatorname{sgn} M_\varepsilon(R) \le k - m, \quad 0 \le |\varepsilon| \ll 1. \tag{4.28}$$

PROOF. Since $e(N_k R^{-1} N_k R) = 2m$, there exists a symplectic matrix $P \in \mathrm{Sp}(2k)$ such that

$$P^{-1}(N_k R^{-1} N_k R)P = Q_1 \diamond Q_2 \tag{4.29}$$

with $\sigma(Q_1) \in \mathbf{U}$, $\sigma(Q_2) \cap \mathbf{U} = \varnothing$, $Q_1 \in \mathrm{Sp}(2m)$, and $Q_2 \in \mathrm{Sp}(2k - 2m)$. By (ii) of Lemma 4.8, since A_3 is invertible we only need to prove (4.28) for $\varepsilon = 0$.

Step 1. Assume A_1 is invertible.

Since R is symplectic, we conclude from $R^\mathsf{T} J_k R = J_k$ that $A_1^\mathsf{T} A_3$ and A_2 are symmetric and

$$A_1^\mathsf{T} A_2 - A_3^\mathsf{T} = I_k.$$

Because R^T is also symplectic, A_1 is symmetric. Hence $A_1 A_3$ is symmetric and

$$A_1 A_2 - A_3^\mathsf{T} = I_k. \tag{4.30}$$

By definition we have

$$\begin{aligned} M_0(R) &= R^\mathsf{T} \begin{pmatrix} 0 & -I_k \\ -I_k & 0 \end{pmatrix} R + \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \\ &= -2 \begin{pmatrix} A_1 A_3 & A_3^\mathsf{T} \\ A_3 & A_2 \end{pmatrix}. \end{aligned} \tag{4.31}$$

Since A_1 is invertible, there holds

$$\begin{aligned} &\begin{pmatrix} I_k & 0 \\ -A_1^{-1} & I_k \end{pmatrix} \begin{pmatrix} A_1 A_3 & A_3^\mathsf{T} \\ A_3 & A_2 \end{pmatrix} \begin{pmatrix} I_k & -A_1^{-1} \\ 0 & I_k \end{pmatrix} \\ &\quad = \begin{pmatrix} A_1 A_3 & 0 \\ 0 & -A_1^{-1} A_3^\mathsf{T} + A_2 \end{pmatrix} \\ &\quad = \begin{pmatrix} A_1 A_3 & 0 \\ 0 & A_1^{-1} \end{pmatrix}, \end{aligned} \tag{4.32}$$

where in the last equality we have used the equality (4.30). From (4.32) we obtain

$$\frac{1}{2} \operatorname{sgn} M_0(R) = -\frac{1}{2} \operatorname{sgn} \begin{pmatrix} A_1 A_3 & 0 \\ 0 & A_1^{-1} \end{pmatrix}. \tag{4.33}$$

By the Jordan canonical form decomposition of a complex matrix, there exists a complex invertible k-order matrix G_1 such that

$$G_1^{-1}A_3G_1 = \begin{pmatrix} u_1 & * & * & * & * \\ 0 & u_2 & * & * & * \\ \vdots & \ddots & \ddots & * & * \\ \vdots & \mathbf{0} & \ddots & u_{k-1} & * \\ 0 & \cdots & \cdots & 0 & u_k \end{pmatrix}$$

with $u_1, u_2, \dots, u_k \in \mathbb{C}$.

(4.2) gives

$$N_k R^{-1} N_k R = I_{2k} + 2\begin{pmatrix} A_3 & A_2 \\ A_1A_3 & A_3^{\mathsf{T}} \end{pmatrix}. \tag{4.34}$$

Since

$$\begin{pmatrix} I_k & 0 \\ -A_1 & I_k \end{pmatrix}\begin{pmatrix} A_3 & A_2 \\ A_1A_3 & A_3^{\mathsf{T}} \end{pmatrix}\begin{pmatrix} I_k & 0 \\ A_1 & I_k \end{pmatrix} = \begin{pmatrix} I_k + 2A_3 & A_2 \\ -A_1 & -I_k \end{pmatrix},$$

by (4.34) we have

$$\begin{pmatrix} I_k & 0 \\ A_1 & I_k \end{pmatrix}^{-1}(N_kR^{-1}N_kR)\begin{pmatrix} I_k & 0 \\ A_1 & I_k \end{pmatrix} = \begin{pmatrix} 3I_k + 4A_3 & 2A_2 \\ -2A_1 & -I_k \end{pmatrix} := R_1. \tag{4.35}$$

By (4.35), for any $\lambda \in \mathbb{C}$ we get

$$\lambda I_{2k} - R_1 = \begin{pmatrix} (\lambda-3)I_k - 4A_3 & -2A_2 \\ 2A_1 & (\lambda+1)I_k \end{pmatrix}. \tag{4.36}$$

Since A_1 is invertible, by (4.30) there holds

$$\begin{aligned} &\begin{pmatrix} I_k & -\frac{1}{2}((\lambda-3)I_k - 4A_3)A_1^{-1} \\ 0 & I_k \end{pmatrix}\begin{pmatrix} (\lambda-3)I_k - 4A_3 & -2A_2 \\ 2A_1 & (\lambda+1)I_k \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & -\frac{1}{2}((\lambda^2-2\lambda+1)I_k - 4\lambda A_3)A_1^{-1} \\ 2A_1 & (\lambda+1)I_k \end{pmatrix}. \end{aligned} \tag{4.37}$$

Then by (4.36)–(4.37) we have

$$\det(\lambda I_{2k} - R_1) = \det((\lambda^2 - 2\lambda + 1)I_k - 4\lambda A_3). \tag{4.38}$$

Denote by $u_1, u_2, \dots, u_k$ the k complex eigenvalues of A_3; (4.38) gives

$$\begin{aligned} \det(\lambda I_{2k} - R_1) &= \prod_{i=1}^{k}(\lambda^2 - 2\lambda + 1 - 4\lambda u_i) \\ &= \prod_{i=1}^{k}(\lambda^2 - (2+4u_i)\lambda + 1). \end{aligned} \tag{4.39}$$

Thus from (4.35) and (4.39) we get

$$\det(\lambda I_{2k} - N_k R^{-1} N_k R) = \prod_{i=1}^{k}(\lambda^2 - 2\lambda + 1 - 4\lambda u_i)$$

$$= \prod_{i=1}^{k}(\lambda^2 - (2 + 4u_i)\lambda + 1). \tag{4.40}$$

It is easy to check that the equation $\lambda^2 - (2+u_i)\lambda + 1 = 0$ has two solutions on $\mathbf{U}$ if and only if $-4 \leq u_i \leq 0$ for $i = 1, \dots, k$. So by (4.29) without loss of generality we assume $u_j \in [-4, 0)$ for $1 \leq j \leq m$ and $u_j \notin [-4, 0)$ for $m + 1 \leq j \leq k$. Then there exists a real invertible matrix of k-order Q such that

$$Q^{-1} A_3 Q = \begin{pmatrix} A_4 & 0 \\ 0 & A_5 \end{pmatrix} := \widetilde{A}_3$$

and $\sigma(A_4) \subset [-4, 0)$, $\sigma(A_5) \cap [-4, 0) = \emptyset$, where A_4 is an m-order real invertible matrix and A_5 is a $(k - m)$-order real matrix.

Denote $\widetilde{A}_1 = Q^{\mathsf{T}} A_1 Q$. We have

$$\widetilde{A}_1 \widetilde{A}_3 = Q^{\mathsf{T}} A_1 Q\, Q^{-1} A_3 Q = Q^{\mathsf{T}} A_1 A_3 Q.$$

Hence both $\widetilde{A}_1$ and $\widetilde{A}_1 \widetilde{A}_3$ are symmetric, and we conclude that

$$\operatorname{sgn} A_1 + \operatorname{sgn}(A_1 A_3) = \operatorname{sgn} \widetilde{A}_1 + \operatorname{sgn}(\widetilde{A}_1 \widetilde{A}_3). \tag{4.41}$$

Correspondingly, we can write $\widetilde{A}_1$ in the block form decomposition

$$\widetilde{A}_1 = \begin{pmatrix} E_1 & E_2 \\ E_2^{\mathsf{T}} & E_4 \end{pmatrix},$$

where E_1 is an m-order real symmetric matrix and E_4 is a $(k - m)$-order real symmetric matrix. Then

$$\widetilde{A}_1 \widetilde{A}_3 = \begin{pmatrix} E_1 A_4 & E_2 A_5 \\ E_2^{\mathsf{T}} A_4 & E_4 A_5 \end{pmatrix}$$

is symmetric.

By the same argument used in the proof of Subcase 2 of Lemma 4.11, without loss of generality we can assume E_1 is invertible (otherwise we can perturb it slightly so that it is invertible). So as in Subcase 1 of the proof of Lemma 4.11, we obtain

$$\begin{pmatrix} I_m & 0 \\ -E_2^{\mathsf{T}} E_1^{-1} & I_{k-m} \end{pmatrix} \begin{pmatrix} E_1 & E_2 \\ E_2^{\mathsf{T}} & E_4 \end{pmatrix} \begin{pmatrix} I_m & -E_1^{-1} E_2 \\ 0 & I_{k-m} \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_4 - E_2^{\mathsf{T}} E_1^{-1} E_2 \end{pmatrix} \tag{4.42}$$

and

$$\begin{pmatrix} I_m & 0 \\ -E_2^{\mathsf{T}}E_1^{-1} & I_{k-m} \end{pmatrix}\begin{pmatrix} E_1A_4 & E_2A_5 \\ E_2^{\mathsf{T}}A_4 & E_4A_5 \end{pmatrix}\begin{pmatrix} I_m & -E_1^{-1}E_2 \\ 0 & I_{k-m} \end{pmatrix} = \begin{pmatrix} E_1A_4 & 0 \\ 0 & (E_4 - E_2^{\mathsf{T}}E_1^{-1}E_2)A_5 \end{pmatrix}. \tag{4.43}$$

By (4.43) we also have E_1A_4 is symmetric. Since E_1 is symmetric and invertible, $\sigma(A_4) \subset [-4, 0)$, by Lemma 4.11 we have

$$\operatorname{sgn}(E_1A_4) + \operatorname{sgn} E_1 = 0. \tag{4.44}$$

By (4.42) and (4.42), there hold

$$\operatorname{sgn} \widetilde{A}_1 = \operatorname{sgn}\big(E_4 - E_2^{\mathsf{T}}E_1^{-1}E_2\big) + \operatorname{sgn} E_1, \tag{4.45}$$

$$\operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) = \operatorname{sgn}\big(\big(E_4 - E_2^{\mathsf{T}}E_1^{-1}E_2\big)A_5\big) + \operatorname{sgn}(E_1A_4). \tag{4.46}$$

(4.44)–(4.46) give

$$\begin{aligned} \operatorname{sgn}(\widetilde{A}_1\widetilde{A}_3) + \operatorname{sgn} \widetilde{A}_1 &= \operatorname{sgn}\big(\big(E_4 - E_2^{\mathsf{T}}E_1^{-1}E_2\big)A_5\big) \\ &\quad + \operatorname{sgn}\big(E_4 - E_2^{\mathsf{T}}E_1^{-1}E_2\big) \\ &\in [-2(k-m), 2(k-m)]. \end{aligned} \tag{4.47}$$

Then (4.28) holds from (4.33), (4.41), and (4.47).

Step 2. Assume A_1 is not invertible.

If $A_1 = 0$, then $A_3 = -I_k$ and $m = k$. It is easy to check that

$$M_0(R) = 2\begin{pmatrix} 0 & I_k \\ I_k & -A_2 \end{pmatrix} \quad \text{is congruent to} \quad 2\begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix},$$

so $\operatorname{sgn} M_0(R) = 0$ and (4.28) holds.

If $1 \le \operatorname{rank} A_1 = r \le k-1$, there is a $k \times k$ invertible matrix G with $\det G > 0$ such that

$$(G^{-1})^{\mathsf{T}}A_1G^{-1} = \operatorname{diag}(0, \Lambda), \tag{4.48}$$

where Λ is a $r \times r$ real invertible matrix. Hence

$$\begin{aligned} \operatorname{diag}((G^{\mathsf{T}})^{-1}, G) \cdot R \cdot \operatorname{diag}(G^{-1}, G^{\mathsf{T}}) &= \begin{pmatrix} (G^{\mathsf{T}})^{-1}A_1G^{-1} & I_k \\ GA_3G^{-1} & GA_2G^{\mathsf{T}} \end{pmatrix} \\ &:= R_2 = \begin{pmatrix} 0 & 0 & I_{k-r} & 0 \\ 0 & \Lambda & 0 & I_r \\ B_1 & B_2 & D_1 & D_2 \\ B_3 & B_4 & D_3 & D_4 \end{pmatrix}, \end{aligned} \tag{4.49}$$

where B_1 and D_1 are $(k-r) \times (k-r)$ matrices and B_4 and D_4 are $r \times r$ matrices.

Since R_2 is symplectic and Λ is invertible, there holds $R_2^{\mathsf{T}} J_k R_2 = J_k$. It implies that $B_3 = 0$, $D_3 = D_2^{\mathsf{T}}$, $B_1 = -I_{k-r}$, and D_1 and D_4 are symmetric. Thus

$$R_2 = \begin{pmatrix} 0 & 0 & I_{k-r} & 0 \\ 0 & \Lambda & 0 & I_r \\ B_1 & B_2 & D_1 & D_2 \\ 0 & B_4 & D_2^{\mathsf{T}} & D_4 \end{pmatrix}.$$

For $t \in [0, 1]$, we define

$$\beta(t) = \begin{pmatrix} 0 & 0 & I_{k-r} & 0 \\ 0 & \Lambda & 0 & I_r \\ B_1 & tB_2 & tD_1 & tD_2 \\ 0 & B_4 & tD_2^{\mathsf{T}} & D_4 \end{pmatrix}.$$

It is easy to check that β is a symplectic path and $\nu_{L_j}(\beta(t)) = 0$ for all $t \in [0, 1]$ and $j = 0, 1$. We also have $\beta(1) = R_2$ and

$$\beta(0) = \begin{pmatrix} 0 & 0 & I_{k-r} & 0 \\ 0 & \Lambda & 0 & I_r \\ B_1 & 0 & 0 & 0 \\ 0 & B_4 & 0 & D_4 \end{pmatrix} = -J_{k-r} \diamond \begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix} := R_3.$$

Then by lemma 2.2 of [32], Lemma 4.9, and Remark 4.3 we have

$$\begin{aligned} \frac{1}{2} \operatorname{sgn} M_0(R_2) &= \frac{1}{2} \operatorname{sgn} M_0(-J_{k-r}) + \frac{1}{2} \operatorname{sgn} M_0 \left(\begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix} \right) \\ &= \frac{1}{2} \operatorname{sgn} M_0 \left(\begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix} \right). \end{aligned} \tag{4.50}$$

Since $R_2 \sim R$, by (4.50) we have

$$\frac{1}{2} \operatorname{sgn} M_0(R) = \frac{1}{2} \operatorname{sgn} M_0 \left(\begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix} \right). \tag{4.51}$$

By (4.2), there holds

$$N_k R_2^{-1} N_k R_2 = I_{2k} + 2 \begin{pmatrix} B_1 & B_2 & D_1 & D_2 \\ 0 & B_4 & D_2^{\mathsf{T}} & D_4 \\ 0 & 0 & B_1^{\mathsf{T}} & 0 \\ 0 & \Lambda B_4 & B_2^{\mathsf{T}} & B_4^{\mathsf{T}} \end{pmatrix}. \tag{4.52}$$

By (4.52) for any $\lambda \in \mathbb{C}$, we obtain

$$\begin{aligned} &\det(\lambda I_{2k} - N_k R_2^{-1} N_k R_2) \\ &\quad = \det((\lambda - 1) I_{k-r} - 2B_1) \det((\lambda - 1) I_{k-r} - 2B_1^{\mathsf{T}}) \\ &\qquad \cdot \det \begin{pmatrix} (\lambda - 1) I_r - 2B_4 & -2D_4 \\ -2\Lambda B_4 & (\lambda - 1) I_r - 2B_4^{\mathsf{T}} \end{pmatrix} \\ &\quad = \det(\lambda I_{2k} - N_k R_3^{-1} N_k R_3), \end{aligned} \tag{4.53}$$

where

$$N_k R_3^{-1} N_k R_3 = I_{2k} + 2 \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_4 & 0 & D_4 \\ 0 & 0 & B_1^{\mathsf{T}} & 0 \\ 0 & \Lambda B_4 & 0 & B_4^{\mathsf{T}} \end{pmatrix}.$$

So (4.53) gives

$$\sigma(N_k R^{-1} N_k R) = \sigma(N_k R_2^{-1} N_k R_2) = \sigma(N_k R_3^{-1} N_k R_3). \tag{4.54}$$

Since $B_1 = -I_{k-r}$ and

$$R_3 = (-J_{k-r}) \diamond \begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix},$$

(4.54) gives

$$e\left(N_r \begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix}^{-1} N_r \begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix}\right) = 2(m - (k - r)). \tag{4.55}$$

Step 1 implies that

$$\frac{1}{2}\left|\operatorname{sgn} M_0 \left(\begin{pmatrix} \Lambda & I_r \\ B_4 & D_4 \end{pmatrix}\right)\right| \le r - (m - (k - r)) = k - m. \tag{4.56}$$

Then (4.28) follows from (4.51) and (4.56). This finishes the proof of Step 2.

With Step 1 and Step 2, the proof of Lemma 4.13 is completed. □

The following result is about the (L_0, L_1)-normal forms of L_0-degenerate symplectic matrices, which generalizes lemma 2.10 of [33].

LEMMA 4.14. *Let $R \in \mathrm{Sp}(2k)$ have the block form*

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \textit{with } 1 \le \operatorname{rank} B = r < k.$$

We have

(i)

$$R \sim \begin{pmatrix} A_1 & B_1 & I_r & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix},$$

where A_1, A_2, A_3 are $r \times r$ matrices, D_1, D_2, D_3 are $(k-r) \times (k-r)$ matrices, B_1, B_3 are $r \times (k-r)$ matrices, and C_2, C_3 are $(k-r) \times r$ matrices.

(ii) *If A_3 is invertible, we have*

$$R \sim \begin{pmatrix} A_1 & I_r \\ A_3 & A_2 \end{pmatrix} \diamond \begin{pmatrix} D_1 & 0 \\ \widetilde{D}_3 & D_2 \end{pmatrix}, \tag{4.57}$$

where $\widetilde{D}_3$ is a $(k-r) \times (k-r)$ matrix.

(iii) *If* $1 \le \operatorname{rank} A_3 = \lambda \le r-1$, *then*

$$R \sim \begin{pmatrix} U & I_\lambda \\ \Lambda & V \end{pmatrix} \diamond \begin{pmatrix} \widetilde{A}_1 & \widetilde{B}_1 & I_{r-\lambda} & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & \widetilde{B}_3 & \widetilde{A}_2 & 0 \\ \widetilde{C}_3 & \widetilde{D}_3 & \widetilde{C}_2 & \widetilde{D}_2 \end{pmatrix}, \tag{4.58}$$

where $\widetilde{A}_1, \widetilde{A}_2$ *are* $(r-\lambda)\times(r-\lambda)$ *matrices,* $\widetilde{B}_1, \widetilde{B}_3$ *are* $(r-\lambda)\times(k-r)$ *matrices,* $\widetilde{C}_2, \widetilde{C}_3$ *are* $(k-r)\times(r-\lambda)$ *matrices,* $D_1, \widetilde{D}_2, \widetilde{D}_3$ *are* $(k-r)\times(k-r)$ *matrices,* U, V, Λ *are* $\lambda\times\lambda$ *matrices, and* Λ *is invertible.*

(iv) *If* $A_3 = 0$, *then* A_1, A_2 *are symmetric and* $A_1A_2 = I_r$. *Suppose* $m^+(A_1) = p$, $m^-(A_1) = r-p$, *and* $0 \le \operatorname{rank} B_3 = \lambda \le \min\{r, k-r\}$, *then*

$$N_k R^{-1} N_k R \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond p+q^-} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond(r-p+q^+)} \diamond I_2^{\diamond q^0} \diamond D(2)^{\diamond\lambda}, \tag{4.59}$$

$$m^+(A^\top C) = \lambda + q^+, \tag{4.60}$$

$$m^0(A^\top C) = r - \lambda + q^0, \tag{4.61}$$

$$m^-(A^\top C) = \lambda + q^-, \tag{4.62}$$

where $q^* \ge 0$ *for* $* = \pm, 0$, $q^+ + q^0 + q^- = k - r - \lambda$, $M^{\diamond 0}$ *means the corresponding component does not appear at all for* M *being one of the four matrices on the right-hand side of* (4.59).

PROOF. By lemma 2.10 of [33] or the same argument used in the proof of theorem 3.1 of [34], (i)–(iii) hold. So we only need to prove (4.59)–(4.62).

By (i) and $A_3 = 0$ we have

$$R \sim \begin{pmatrix} A_1 & B_1 & I_r & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix} := R_1. \tag{4.63}$$

Since R_1 is symplectic we have $R_1^\top J_k R_1 = J_k$. Then we have A_1, A_2 are symmetric and $A_1A_2 = I_r$. $D_1 D_2^\top = I_{k-r}$ and $A_1^\top B_3 = C_3^\top D_1$. (4.2) yields

$$N_k R_1^{-1} N_k R_1 = \begin{pmatrix} I_r & 2B_3 & 2A_2 & 0 \\ 0 & I_{k-r} & 0 & 0 \\ 0 & 2A_1^\top B_3 & I_r & 0 \\ 2B_3^\top A_1 & 2B_1^\top B_3 + 2D_1^\top D_3 & 2B_3^\top & I_{k-r} \end{pmatrix}. \tag{4.64}$$

By Remark 4.6 we obtain

$$m^*(A^\top C) = m^*\left(\begin{pmatrix} 0 & A_1^\top B_3 \\ B_3^\top A_1 & B_1^\top B_3 + D_1^\top D_3 \end{pmatrix}\right), \quad * = +, -, 0. \tag{4.65}$$

Since $0 \le \operatorname{rank} B_3 = \lambda \le \min\{r,\ k-r\}$, there exist $r \times r$ and $(k-r)\times(k-r)$ real invertible matrices G_1 and G_2 such that

$$G_1 B_3 G_2 = \begin{pmatrix} I_\lambda & 0 \\ 0 & 0 \end{pmatrix} := F. \tag{4.66}$$

Note that if $\lambda = 0$ then $B_3 = 0$l if $\lambda = \min\{r,\ k-r\}$ then

$$G_1 B_3 G_2 = \begin{pmatrix} I_\lambda & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} cI_\lambda \\ 0 \end{pmatrix};$$

if $\lambda = r = k - r$ then $G_1 B_3 G_2 = I_\lambda$. The proof below can still go through by a suitable adjustment.

By (4.66) we have

$$\begin{aligned} &\begin{pmatrix} G_1 A_1^{-1} & 0 \\ 0 & G_2^\top \end{pmatrix} \begin{pmatrix} 0 & A_1^\top B_3 \\ B_3^\top A_1 & B_1^\top B_3 + D_1^\top D_3 \end{pmatrix} \begin{pmatrix} A_1^{-1} G_1^\top & 0 \\ 0 & G_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & G_1 B_3 G_2 \\ G_2^\top B_3^\top G_1^\top & U \end{pmatrix} = \begin{pmatrix} 0 & 0 & I_\lambda & 0 \\ 0 & 0 & 0 & 0 \\ I_\lambda & 0 & U_1 & U_2 \\ 0 & 0 & U_2^\top & U_4 \end{pmatrix}. \end{aligned} \tag{4.67}$$

Then

$$\begin{aligned} &\begin{pmatrix} I_\lambda & 0 & 0 & 0 \\ 0 & I_{r-\lambda} & 0 & 0 \\ -\frac{1}{2}U_1 & 0 & I_\lambda & 0 \\ -U_2^\top & 0 & 0 & I_{k-r-\lambda} \end{pmatrix} \begin{pmatrix} 0 & 0 & I_\lambda & 0 \\ 0 & 0 & 0 & 0 \\ I_\lambda & 0 & U_1 & U_2 \\ 0 & 0 & U_2^\top & U_4 \end{pmatrix} \\ &\cdot \begin{pmatrix} I_\lambda & 0 & -\frac{1}{2}U_1 & -U_2 \\ 0 & I_{r-\lambda} & 0 & 0 \\ 0 & 0 & I_\lambda & 0 \\ 0 & 0 & 0 & I_{k-r-\lambda} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & I_\lambda & 0 \\ 0 & 0 & 0 & 0 \\ I_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & U_4 \end{pmatrix}. \end{aligned} \tag{4.68}$$

Set

$$q^* = m^*(U_4), \quad * = \pm,\ 0. \tag{4.69}$$

Then $q^+ + q^0 + q^- = k - r - \lambda$ and (4.60)–(4.62) hold from (4.65), (4.67), and (4.68).

Also by (4.68) and Lemma 4.4 we have

$$\begin{pmatrix} I_{k-r-\lambda} & 0 \\ 2U_4 & I_{k-r-\lambda} \end{pmatrix} \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond q^-} \diamond I_2^{\diamond q^0} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond q^+}. \tag{4.70}$$

By (4.67), there holds

$$
\begin{aligned}
&\operatorname{diag}((G_1^\top)^{-1}A_1, G_2^{-1}, G_1A_1^{-1}, G_2^\top)(N_kR_1^{-1}N_kR_1)\\
&\quad\cdot \operatorname{diag}(A_1^{-1}G_1^\top, G_2, A_1G_1^{-1}, (G_2^\top)^{-1})\\
&= \begin{pmatrix} I_r & 2E & 2\widetilde{A}_1 & 0\\ 0 & I_{k-r} & 0 & 0\\ 0 & 2F & I_r & 0\\ 2F^\top & 2U & 2E^\top & I_{k-r}\end{pmatrix} := M,
\end{aligned}
\tag{4.71}
$$

where $\widetilde{A}_1 = (G_1^\top)^{-1}A_1G_1^{-1}$ and $E = (G_1^\top)^{-1}A_1B_3G_2 = \widetilde{A}_1F$.

Since M is symplectic, we have $M^\top J_kM = J_k$. Then we have $E = \widetilde{A}_1F$. Since $\widetilde{A}_1 = (G_1^\top)^{-1}A_1G_1^{-1}$, it is congruent to $\operatorname{diag}(a_1, a_2, \dots, a_r)$ with

$$
\begin{aligned}
a_i &= 1, \quad 1 \le i \le p,\\
a_j &= -1, \quad p+1 \le j \le r \text{ for some } 0 \le p \le r.
\end{aligned}
\tag{4.72}
$$

Then there is an invertible $r \times r$ real matrix Q such that $\det Q > 0$ and

$$
\begin{aligned}
Q\widetilde{A}_1Q^\top &= \operatorname{diag}(a_1, a_2, \dots, a_r)\\
&= \operatorname{diag}(\operatorname{diag}(a_1, a_2, \dots, a_\lambda), \operatorname{diag}(a_{\lambda+1}, \dots, a_r))\\
&:= \operatorname{diag}(\Lambda_1, \Lambda_2).
\end{aligned}
\tag{4.73}
$$

Since $\det Q > 0$ we can join it to I_r by an invertible continuous matrix path. So there is a continuous invertible symmetric matrix path α_1 such that $\alpha_1(1) = \widetilde{A}_1$ and $\alpha_1(0) = \operatorname{diag}(a_1, a_2, \dots, a_r)$ with

$$
m^*(\alpha_1(t)) = m^*(\widetilde{A}_1) = m^*(A_1), \quad t \in [0,1],\ * = +, -.
$$

Define symmetric matrix path

$$
\alpha_2(t) = \begin{pmatrix} 2tU_1 & 2tU_2\\ 2tU_2^\top & 2U_4\end{pmatrix}, \quad t \in [0,1].
$$

For $t \in [0,1]$, define

$$
\beta(t) = \begin{pmatrix} I_r & 2\alpha_1(t)F & 2\alpha_1(t) & 0\\ 0 & I_{k-r} & 0 & 0\\ 0 & 2F & I_r & 0\\ 2F^\top & \alpha_2(t) & 2F^\top\alpha_1(t)^\top & I_{k-r}\end{pmatrix}.
$$

Then since M is symplectic, it is easy to check that β is a continuous path of symplectic matrices. Since

$$
F = \begin{pmatrix} I_\lambda & 0\\ 0 & 0\end{pmatrix}
$$

and $\alpha_1(t)$ is invertible, by direct computation, we have

$$
\begin{aligned}
\operatorname{rank}(\beta(t) - I_{2k}) &= 2\lambda + \operatorname{rank}(\alpha_1(t)) + \operatorname{rank}(U_4)\\
&= 2\lambda + r + m^+(U_4) + m^-(U_4).
\end{aligned}
$$

Hence

$$\nu_1(\beta(t)) = \nu_1(\beta(1)) = \nu_1(M), \quad t \in [0,1].$$

Because $\sigma(\beta(t)) = \{1\}$, by Definition 2.2 and Lemma 2.4

$$\begin{aligned}
M &= \beta(1) \approx \beta(0) \\
&= \begin{pmatrix} I_\lambda & 0 & 2\Lambda_1 & 2\Lambda_1 & 0 & 0 \\ 0 & I_{r-\lambda} & 0 & 0 & 2\Lambda_2 & 0 \\ 0 & 0 & I_\lambda & 0 & 0 & 0 \\ 0 & 0 & 2I_\lambda & I_\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{r-\lambda} & 0 \\ 2I_\lambda & 0 & 0 & 2\Lambda_1 & 0 & I_\lambda \end{pmatrix} \diamond \begin{pmatrix} I_{k-r-\lambda} & 0 \\ 2U_4 & I_{k-r-\lambda} \end{pmatrix} \\
&\approx \begin{pmatrix} I_\lambda & 2\Lambda_1 & 2\Lambda_1 & 0 \\ 0 & I_\lambda & 0 & 0 \\ 0 & 2I_\lambda & I_\lambda & 0 \\ 2I_\lambda & 0 & 2\Lambda_1 & I_\lambda \end{pmatrix} \diamond \begin{pmatrix} I_{r-\lambda} & 2\Lambda_2 \\ 0 & I_{r-\lambda} \end{pmatrix} \diamond \begin{pmatrix} I_{k-r-\lambda} & 0 \\ 2U_4 & I_{k-r-\lambda} \end{pmatrix} \\
&= \begin{pmatrix} I_\lambda & 2\Lambda_1 & 2\Lambda_1 & 0 \\ 0 & I_\lambda & 0 & 0 \\ 0 & 2I_\lambda & I_\lambda & 0 \\ 2I_\lambda & 0 & 2\Lambda_1 & I_\lambda \end{pmatrix} \diamond \diamond_{j=\lambda+1}^{r} \begin{pmatrix} 1 & 2a_j \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} I_{k-r-\lambda} & 0 \\ 2U_4 & I_{k-r-\lambda} \end{pmatrix}.
\end{aligned}$$

We define the continuous symplectic matrix path

$$\psi(t) = \begin{pmatrix} I_\lambda & 2(1-t^2)\Lambda_1 & 2\Lambda_1 & 0 \\ 0 & (1+t)I_\lambda & 0 & 0 \\ 0 & 2(1-t^2)I_\lambda & I_\lambda & 0 \\ 2(1-t)I_\lambda & 0 & 2(1-t)\Lambda_1 & \frac{1}{1+t}I_\lambda \end{pmatrix}, \quad t \in [0,1].$$

Since Λ_1 is invertible, $\nu(\psi(t)) \equiv \lambda$ for $t \in [0,1]$. So by $\sigma(\psi(t)) \cap \mathbf{U} = \{1\}$ for $t \in [0,t]$ and Definition 2.2 we obtain

$$\begin{aligned}
\begin{pmatrix} I_\lambda & \Lambda_1 & 2\Lambda_1 & 0 \\ 0 & I_\lambda & 0 & 0 \\ 0 & 2I_\lambda & I_\lambda & 0 \\ 2I_\lambda & 0 & 2\Lambda_1 & I_\lambda \end{pmatrix} &= \psi(0) \approx \psi(1) \\
&= \begin{pmatrix} I_\lambda & 2\Lambda_1 \\ 0 & I_\lambda \end{pmatrix} \diamond \begin{pmatrix} 2I_\lambda & 0 \\ 0 & \frac{1}{2}I_\lambda \end{pmatrix} \\
&= \diamond_{j=1}^{\lambda} \begin{pmatrix} 1 & 2a_j \\ 0 & 1 \end{pmatrix} \diamond D(2)^{\diamond\lambda}. \qquad (4.74)
\end{aligned}$$

Thus by (4.74), (4.74), and Remark 2.3 we get

$$M \approx \left(\diamond_{j=1}^{r} \begin{pmatrix} 1 & a_j \\ 0 & 1 \end{pmatrix} \right) \diamond D(2)^{\diamond\lambda} \diamond \begin{pmatrix} I_{k-r-\lambda} & 0 \\ U_4 & I_{k-r-\lambda} \end{pmatrix}. \qquad (4.75)$$

So by (4.70), (4.72), and Remark 2.3, there holds

$$M \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond(p+q^-)} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond(r-p+q^+)} \diamond I_2^{\diamond q^0} \diamond D(2)^{\diamond\lambda}. \tag{4.76}$$

By Lemma 4.7, (4.63), and (4.71), we have

$$N_k R^{-1} N_k R \approx M. \tag{4.77}$$

Then (4.59) holds from (4.76) and (4.77). The proof of Lemma 4.14 is completed. □

5 The Mixed (L_0, L_1)-Concavity

DEFINITION 5.1. The mixed (L_0, L_1)-concavity and mixed (L_1, L_0)-concavity of a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ are defined respectively by

$$\mu_{(L_0,L_1)}(\gamma) = i_{L_0}(\gamma) - \nu_{L_1}(\gamma), \quad \mu_{(L_1,L_0)}(\gamma) = i_{L_1}(\gamma) - \nu_{L_0}(\gamma).$$

Proposition C of [21], proposition 6.1 of [17], and Theorem 4.2 imply the following result:

PROPOSITION 5.2. *There hold*

$$\mu_{(L_0,L_1)}(\gamma) + \mu_{(L_1,L_0)}(\gamma) = i(\gamma^2) - \nu(\gamma^2) - n, \tag{5.1}$$

$$\mu_{(L_0,L_1)}(\gamma) - \mu_{(L_1,L_0)}(\gamma) = \mathrm{concav}^*_{(L_0,L_1)}(\gamma) = \frac{1}{2}\,\mathrm{sgn}\, M_\varepsilon(\gamma(\tau)), \quad 0 < -\varepsilon \ll 1. \tag{5.2}$$

Theorem 1.10 in Section1 is a special case of the following result:

THEOREM 5.3. *For $\gamma \in \mathcal{P}_\tau(2n)$, let $P = \gamma(\tau)$. If $i_{L_0}(\gamma) \geq 0$, $i_{L_1}(\gamma) \geq 0$, $i(\gamma) \geq n$, and $\gamma^2(t) = \gamma(t-\tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$, then*

$$\mu_{(L_0,L_1)}(\gamma) + S^+_{P^2}(1) \geq 0, \tag{5.3}$$

$$\mu_{(L_1,L_0)}(\gamma) + S^+_{P^2}(1) \geq 0. \tag{5.4}$$

PROOF. The proofs of (5.3) and (5.4) are almost the same. We only prove (5.4), which yields Theorem 1.10.

Claim 5.4. Under the conditions of Theorem 5.3, if

$$P^2 \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond p_1} \diamond D(2)^{\diamond p_2} \diamond \widetilde{P}, \tag{5.5}$$

then

$$i(\gamma^2) + 2S^+_{P^2}(1) - \nu(\gamma^2) \geq n + p_1 + p_2. \tag{5.6}$$

PROOF OF CLAIM 5.4. By theorem 7.8 of [19] we have

$$
\begin{aligned}
P \approx\ & I_2^{\diamond q_1} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond q_2} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond q_3} \diamond (-I_2)^{\diamond q_4} \\
& \diamond \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}^{\diamond q_5} \diamond \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}^{\diamond q_6} \\
& \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_{q_7}) \diamond \cdots \diamond R(\theta_{q_7+q_8}) \\
& \diamond N_2(\omega_1, b_1) \diamond \cdots \diamond N_2(\omega_{q_9}, b_{q_9}) \\
& \diamond D(2)^{\diamond q_{10}} \diamond D(-2)^{\diamond q_{11}},
\end{aligned} \tag{5.7}
$$

where $q_i \geq 0$ for $1 \leq i \leq 11$ with $q_1 + q_2 + \cdots + q_8 + 2q_9 + q_{10} + q_{11} = n$, $\theta_j \in (0, \pi)$ for $1 \leq j \leq q_7$, $\theta_j \in (\pi, 2\pi)$ for $q_7 + 1 \leq j \leq q_7 + q_8$, $\omega_j \in (\mathbf{U} \setminus \mathbb{R})$ for $1 \leq j \leq q_9$, and

$$
b_j = \begin{pmatrix} b_{j1} & b_{j2} \\ b_{j3} & b_{j4} \end{pmatrix} \quad \text{satisfying } b_{j2} \neq b_{j3} \text{ for } 1 \leq j \leq q_9.
$$

By (5.7) and Remark 2.3 we obtain

$$
\begin{aligned}
P^2 \approx\ & I_2^{\diamond(q_1+q_4)} \diamond \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond(q_2+q_6)} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond(q_3+q_5)} \\
& \diamond R(2\theta_1) \diamond \cdots \diamond R(2\theta_{q_7}) \diamond \cdots \diamond R(2\theta_{q_7+q_8}) \\
& \diamond N_2(\omega_1, b_1)^2 \diamond \cdots \diamond N_2(\omega_{q_9}, b_{q_9})^2 \diamond D(2)^{\diamond(q_{10}+q_{11})}.
\end{aligned} \tag{5.8}
$$

By theorem 7.8 of [19], (5.5), and (5.8), there hold

$$
q_2 + q_6 \geq p_1, \quad q_{10} + q_{11} \geq p_2. \tag{5.9}
$$

Since $\gamma^2(t) = \gamma(t - \tau)\gamma(\tau)$ for all $t \in [\tau, 2\tau]$, we have γ^2 is also the second iteration of γ in the periodic boundary value case, so by the Bott-type formula (cf. theorem 9.2.1 of [20]), the proof of lemma 4.1 of [21], and Lemma 2.5, we have

$$
\begin{aligned}
& i(\gamma^2) + 2S_{P^2}^+(1) - \nu(\gamma^2) \\
&= 2i(\gamma) + 2S_P^+(1) + \sum_{\theta \in (0,\pi)} \big(S_P^+(e^{\sqrt{-1}\theta}) \\
&\quad - \Big(\sum_{\theta \in (0,\pi)} S_P^-(e^{\sqrt{-1}\theta}) + (\nu(P) - S_P^+(1)) + (\nu_{-1}(P) - S_P^-(-1))\Big) \\
&= 2i(\gamma) + 2(q_1 + q_2) + (q_8 - q_7) - (q_1 + q_3 + q_4 + q_5) \\
&\geq 2n + q_1 + 2q_2 + (q_8 - q_7) - (q_3 + q_4 + q_5) \\
&= n + (2q_1 + 3q_2 + q_6 + 2q_8 + 2q_9 + q_{10} + q_{11}) \\
&\geq n + 2q_2 + q_6 + q_{10} + q_{11} \\
&\geq n + p_1 + p_2,
\end{aligned} \tag{5.10}
$$

where in the first equality we have used $S^+_{P^2}(1) = S^+_P(1) + S^+_P(-1)$ and $\nu(\gamma^2) = \nu(\gamma) + \nu_{-1}(\gamma)$, in the first inequality we have used the condition $i(\gamma) \geq n$, in the third equality we have used that $q_1 + q_2 + \cdots + q_8 + 2q_9 + q_{10} + q_{11} = n$, and in the last inequality we have used (5.9). By (5.10) Claim 5.4 holds. □

We continue with the proof of Theorem 5.3. We set $\mathcal{A} = \mu_{(L_1,L_0)}(\gamma) + S^+_{P^2}(1)$ and $\mathcal{B} = \mu_{(L_0,L_1)}(\gamma) + S^+_{P^2}(1)$.

By proposition C of [21] and proposition 6.1 of [17] we have

$$i_{L_0}(\gamma) + i_{L_1}(\gamma) = i(\gamma^2) - n, \quad \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma) = \nu(\gamma^2). \tag{5.11}$$

From (5.11) or (5.1) we obtain

$$\mathcal{A} + \mathcal{B} = i(\gamma^2) + 2S^+_{P^2}(1) - \nu(\gamma^2) - n. \tag{5.12}$$

Case 1. $\nu_{L_0}(\gamma) = 0$.

In this case, $i_{L_1}(\gamma) + S^+_{P^2}(1) - \nu_{L_0}(\gamma) \geq 0 + 0 - 0 = 0$ and (5.4) holds.

Case 2. $\nu_{L_0}(\gamma) = n$.

In this case

$$P = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

so A is invertible and

$$m^0(A^\mathsf{T} C) = \nu_{L_1}(P) = \nu_{L_1}(\gamma). \tag{5.13}$$

By Lemma 4.4 we have

$$\begin{aligned} NP^{-1}NP &= \begin{pmatrix} I_n & 0 \\ 2A^\mathsf{T} C & I_n \end{pmatrix} \\ &\approx I_2^{\diamond m^0(A^\mathsf{T} C)} \diamond N_1(1,1)^{\diamond m^-(A^\mathsf{T} C)} \diamond N_1(1,-1)^{\diamond m^+(A^\mathsf{T} C)}. \end{aligned} \tag{5.14}$$

By Claim 5.4, (5.14), and (5.12), there holds

$$\mathcal{A} + \mathcal{B} \geq m^-(A^\mathsf{T} C). \tag{5.15}$$

By Theorem 4.2, Lemma 4.8, and (5.13) we obtain

$$\mathcal{A} - \mathcal{B} \geq m^+(A^\mathsf{T} C) + m^0(A^\mathsf{T} C) - n. \tag{5.16}$$

Then (5.15) and (5.16) give

$$2\mathcal{A} \geq m^-(A^\mathsf{T} C) + (m^+(A^\mathsf{T} C) + m^0(A^\mathsf{T} C)) - n = 0,$$

which yields $\mathcal{A} \geq 0$ and (5.4) holds.

Case 3. $1 \leq \nu_{L_0}(\gamma) = \nu_{L_0}(P) \leq n - 1$.

In this case by (i) of Lemma 4.14 we have

$$P := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim \begin{pmatrix} A_1 & B_1 & I_r & 0 \\ 0 & D_1 & 0 & 0 \\ A_3 & B_3 & A_2 & 0 \\ C_3 & D_3 & C_2 & D_2 \end{pmatrix},$$

where A_1, A_2, A_3 are $r \times r$ matrices, D_1, D_2, D_3 are $(n-r)\times(n-r)$ matrices, B_1, B_3 are $r\times(n-r)$ matrices, and C_2, C_3 are $(n-r)\times r$ matrices. We divide Case 3 into the following three subcases.

SUBCASE 1. $A_3 = 0$.

In this subcase let $\lambda = \operatorname{rank} B_3$. Then $0 \le \lambda \le \min\{r, n-r\}$, A_1 is invertible, $A_1A_2 = I_r$, and $D_1D_2^{\mathsf T} = I_{k-r}$, so we have A is invertible; furthermore, there holds $m^0(A^{\mathsf T}C) = \dim\ker C = \nu_{L_1}(P)$. Suppose $m^+(A_1) = p$, $m^-(A_1) = r-p$; then by (iv) of Lemma 4.14 we have

$$N_kR^{-1}N_kR \approx \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{\diamond p+q^-} \diamond \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}^{\diamond(r-p+q^+)} \diamond I_2^{\diamond q^0} \diamond D(2)^{\diamond\lambda}, \tag{5.17}$$

$$m^+(A^{\mathsf T}C) = \lambda + q^+, \tag{5.18}$$

$$m^0(A^{\mathsf T}C) = r - \lambda + q^0, \tag{5.19}$$

$$m^-(A^{\mathsf T}C) = \lambda + q^-, \tag{5.20}$$

where $q^* \ge 0$ for $* = +, -, 0$ and $q^+ + q^0 + q^- = n - r - \lambda$.

By (5.17) and Claim 5.4, there holds

$$i(\gamma^2) + 2S^+_{P^2}(1) - \nu(\gamma^2) \ge n + p + q^- + \lambda \ge n + q^- + \lambda. \tag{5.21}$$

(5.21) and (5.12) give

$$\mathcal{A} + \mathcal{B} \ge q^- + \lambda. \tag{5.22}$$

By Theorem 4.2, Lemma 4.8, and (5.18)–(5.20), we have

$$\begin{aligned}\mathcal{A} - \mathcal{B} &\ge m^+(A^{\mathsf T}C) + m^0(A^{\mathsf T}C) - n = q^+ + \lambda + r - \lambda + q^0 - n\\ &= r + q^+ + q^0 - n.\end{aligned} \tag{5.23}$$

Since $q^+ + q^0 + q^- = n - r - \lambda$, (5.22) and (5.23) imply

$$\begin{aligned}2\mathcal{A} &\ge q^- + \lambda + (r + q^+ + q^0) - n\\ &= (q^- + q^+ + q^0) - (n - r - \lambda)\\ &= 0,\end{aligned}$$

which yields (5.4).

SUBCASE 2. A_3 is invertible.

In this case by (ii) of Lemma 4.14 there holds

$$P \sim \begin{pmatrix}A_1 & I_r\\ A_3 & A_2\end{pmatrix} \diamond \begin{pmatrix}D_1 & 0\\ \widetilde{D}_3 & D_2\end{pmatrix} := P_1 \diamond P_2, \tag{5.24}$$

where $\widetilde{D}_3$ is a $(k-r)\times(k-r)$ matrix. Then by (5.24) and Lemma 4.7 we obtain

$$P^2 \approx (N_rP_1^{-1}N_rP_1) \diamond (N_{n-r}P_2^{-1}N_{n-r}P_2). \tag{5.25}$$

Let $e(N_r P_1^{-1} N_r P_1) = 2m$; by Lemma 4.13 we have $0 \le m \le r$ and

$$\frac{1}{2}\operatorname{sgn} M_\varepsilon(P_1) \le r - m, \quad 0 < -\varepsilon \ll 1. \tag{5.26}$$

Also by (5.25) and (5.8), there exists $\widetilde{P}_1 \in \mathrm{Sp}(2m)$ such that

$$N_r P_1^{-1} N_r P_1 \approx D(2)^{\diamond(r-m)} \diamond \widetilde{P}_1. \tag{5.27}$$

By Lemma 4.4, there holds

$$\begin{aligned} N_{n-r} P_2^{-1} N_{n-r} P_2 &= \begin{pmatrix} I_{n-r} & 0 \\ 2D_1^{\mathsf{T}}\widetilde{D}_3 & I_{n-r} \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond m^-(D_1^{\mathsf{T}}\widetilde{D}_3)} \diamond I_2^{\diamond m^0(D_1^{\mathsf{T}}\widetilde{D}_3)} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond m^+(D_1^{\mathsf{T}}\widetilde{D}_3)}. \end{aligned} \tag{5.28}$$

So by Claim 5.4 and (5.27), (5.28), (5.25), and (5.12) we have

$$\mathcal{A} + \mathcal{B} \ge m^-(D_1^{\mathsf{T}}\widetilde{D}_3) + r - m. \tag{5.29}$$

By Theorem 4.2 and Lemma 4.8 together with Lemma 4.13, for $0 < -\varepsilon \ll 1$ we get

$$\begin{aligned} \mathcal{A} - \mathcal{B} &= -\frac{1}{2}\operatorname{sgn} M_\varepsilon(P_1) - \frac{1}{2}\operatorname{sgn} M_\varepsilon(P_2) \\ &\ge -r + m - (n - r) + m^+(D_1^{\mathsf{T}}\widetilde{D}_3) + m^0(D_1^{\mathsf{T}}\widetilde{D}_3) \\ &= m + m^+(D_1^{\mathsf{T}}\widetilde{D}_3) + m^0(D_1^{\mathsf{T}}\widetilde{D}_3) - n, \end{aligned} \tag{5.30}$$

where we have used the fact that $m^0(D_1^{\mathsf{T}}\widetilde{D}_3) = \ker(\widetilde{D}_3) = \nu_{L_1}(P_2)$.

Note that

$$m^+(D_1^{\mathsf{T}}\widetilde{D}_3) + m^0(D_1^{\mathsf{T}}\widetilde{D}_3) + m^-(D_1^{\mathsf{T}}\widetilde{D}_3) = n - r. \tag{5.31}$$

Then by (5.29), (5.30), and (5.31) we have

$$\begin{aligned} 2\mathcal{A} &\ge m^-(D_1^{\mathsf{T}}\widetilde{D}_3) + r - m + (m + m^+(D_1^{\mathsf{T}}\widetilde{D}_3) + m^0(D_1^{\mathsf{T}}\widetilde{D}_3)) - n \\ &= m^+(D_1^{\mathsf{T}}\widetilde{D}_3) + m^0(D_1^{\mathsf{T}}\widetilde{D}_3) + m^-(D_1^{\mathsf{T}}\widetilde{D}_3) - (n - r) \\ &= 0, \end{aligned}$$

which yields (5.4).

SUBCASE 3. $1 \le \operatorname{rank} A_3 = l \le r - 1$.

In this case by (iii) of Lemma 4.14 there holds

$$P \sim \begin{pmatrix} U & I_l \\ \Lambda & V \end{pmatrix} \diamond \begin{pmatrix} \widetilde{A}_1 & \widetilde{B}_1 & I_{r-l} & 0 \\ 0 & D_1 & 0 & 0 \\ 0 & \widetilde{B}_3 & \widetilde{A}_2 & 0 \\ \widetilde{C}_3 & \widetilde{D}_3 & \widetilde{C}_2 & \widetilde{D}_2 \end{pmatrix} := P_3 \diamond P_4, \tag{5.32}$$

where $\widetilde{A}_1, \widetilde{A}_2$ are $(r-l)\times(r-l)$ matrices, $\widetilde{B}_1, \widetilde{B}_3$ are $(r-l)\times(n-r)$ matrices, $\widetilde{C}_2, \widetilde{C}_3$ are $(n-r)\times(r-l)$ matrices, $D_1, \widetilde{D}_2, \widetilde{D}_3$ are $(n-r)\times(n-r)$ matrices, U, V, Λ are $l\times l$ matrices, and Λ is invertible.

Let $\lambda = \operatorname{rank} \widetilde{B}_3$ and denote

$$P_4 = \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix},$$

where $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ are $(n-l)$-order real matrices. Assume $m^+(\widetilde{A}_1) = p$ and $m^-(\widetilde{A}_1) = r-l-p$; then by (iv) of Lemma 4.14 we have

$$N_k P_4^{-1} N_k P_4 \approx \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\diamond(p+q^-)} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{\diamond(r-l-p+q^+)} \diamond I_2^{\diamond q^0} \diamond D(2)^{\diamond\lambda}, \tag{5.33}$$

$$m^+(\widetilde{A}^\top \widetilde{C}) = \lambda + q^+, \tag{5.34}$$

$$m^0(\widetilde{A}^\top \widetilde{C}) = r-l-\lambda + q^0, \tag{5.35}$$

$$m^-(\widetilde{A}^\top \widetilde{C}) = \lambda + q^-, \tag{5.36}$$

where $q^* \geq 0$ for $* = +,-,0$ and $q^+ + q^0 + q^- = n-r-\lambda$.

Let $e(N_l P_3^{-1} N_l P_3) = 2m$. By Lemma 4.13 we obtain $0 \leq m \leq l$ and

$$\frac{1}{2}\operatorname{sgn} M_\varepsilon(P_3) \leq l-m, \quad 0 < -\varepsilon \ll 1. \tag{5.37}$$

By similar argument as in the proof of Subcase 2, there exists $\widetilde{P}_3 \in \mathrm{Sp}(2m)$ such that

$$N_r P_3^{-1} N_r P_3 \approx D(2)^{\diamond(l-m)} \diamond \widetilde{P}_3. \tag{5.38}$$

So by Claim 5.4, (5.32), (5.33), (5.38), and (5.12) we have

$$\mathcal{A} + \mathcal{B} \geq q^- + l - m + \lambda. \tag{5.39}$$

By Theorem 4.2, Lemma 4.8, (5.34), (5.35) and (5.37), for $0 \leq -\varepsilon \ll 1$ we obtain

$$\begin{aligned} \mathcal{A} - \mathcal{B} &= -\frac{1}{2}\operatorname{sgn} M_\varepsilon(P_3) - \frac{1}{2}\operatorname{sgn} M_\varepsilon(P_4) \\ &\geq -\frac{1}{2}\operatorname{sgn} M_\varepsilon(P_3) - (n-l) - m^+(\widetilde{A}^\top\widetilde{C}) + m^0(\widetilde{A}^\top\widetilde{C}) \\ &\geq -l + m - (n-l) + (\lambda + q^+) + (r-l-\lambda+q^0) \\ &= (q^+ + q^0 + r) - n - (l-m). \end{aligned} \tag{5.40}$$

Since $q^+ + q^0 + q^- = n - r - \lambda$, by (5.39) and (5.40) we have

$$\begin{aligned}2\mathcal{A} &\geq q^- + l - m + \lambda + (q^+ + q^0 + r) - n - (l - m)\\ &= (q^+ + q^0 + q^-) - (n - r - \lambda)\\ &= 0,\end{aligned}$$

which yields (5.4). Hence (5.4) holds in Cases 1 through 3 and the proof of Theorem 5.3 is completed. □

Remark 5.5. Both the estimates (5.3) and (5.4) in Theorem 5.3 are optimal . In fact, we can construct a symplectic path satisfying the conditions of Theorem 5.3 such that the equalities in (5.3) and (5.4) hold. Let $\tau = \pi$ and $\gamma(t) = R(t)^{\diamond n}$, $t \in [0, \pi]$. It is easy to see that

$$i_{L_0}(\gamma) = \sum_{0<t<\pi} \nu_{L_0}(\gamma(t)) = 0 \quad \text{and also} \quad i_{L_1}(\gamma) = \sum_{0<t<\pi} \nu_{L_1}(\gamma(t)) = 0,$$

$\nu_{L_0}(\gamma) = \nu_{L_1}(\gamma) = n$, $\gamma^2(t) = \gamma(t-\pi)\gamma(\pi)$ for $t \in [\pi, 2\pi]$, $i(\gamma) = n$, and $P = \gamma(\pi) = -I_{2n}$. Hence by Lemma 2.5, $S^+_{P^2}(1) = S^+_{I_{2n}}(1) = n$. Thus

$$\mu_{(L_0,L_1)}(\gamma) + S^+_{P^2}(1) = \mu_{(L_1,L_0)}(\gamma) + S^+_{P^2}(1) = 0 - n + n = 0.$$

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