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CONSTRUCTION OF ORTHOGONAL AND NEARLY ORTHOGONAL LATIN HYPERCUBE DESIGNS FROM ORTHOGONAL DESIGNS

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Abstract: Latin hypercube designs (LHDs), widely used for computer experiments, are a very large class of designs with desirable properties. Recently, a number of methods have been proposed to construct orthogonal LHDs. In this paper, we introduce an approach to constructing 2^r-order orthogonal designs. The methods are simple and easy to implement. Using orthogonal designs, we propose some methods for constructing orthogonal and nearly orthogonal LHDs so that the elementwise square of each column and the elementwise product of any two distinct columns are orthogonal to all columns. Further, the resulting nearly orthogonal LHDs with $2^{r+1}+2$ runs and 2^r factors have the minimum correlation between any two distinct columns.

Key words and phrases: Computer experiment, correlation, Latin hypercube design, orthogonal design, orthogonality.

1. Introduction

Computer experiments are increasingly popular surrogates for physical experiments. Latin hypercube designs (LHDs), introduced by McKay, Beckman, and Conover (1979), are used for computer experiments that are mostly deterministic. An LHD with n runs and m factors is denoted by a matrix $L(n, m) = (l_1, \ldots, l_m)$, where l_j is the jth factor, and each factor includes n uniformly spaced levels. An LHD is called an orthogonal LHD if the inner product of any two distinct columns of this LHD is zero. Note that we are defining orthogonality as zero inner product. In the past two decades, a lot of work has been done to construct orthogonal LHDs with appealing properties. Ye (1998) proposed a method to construct a class of orthogonal LHDs with $n = 2^{r+1} + 1$ runs and $m = 2r$ factors ($r = 1, 2, \ldots$) using permutation matrices. Cioppa and Lucas (2007) extended Ye's approach by adding new orthogonal columns to his orthogonal LHDs. The numbers of factors in the orthogonal LHDs of Cioppa and Lucas (2007) can be as large as $1 + r + \binom{r}{2}$. Steinberg and Lin (2006) and Pang, Liu, and Lin (2009) constructed orthogonal LHDs with n runs, where $n = p^d$, $d = 2^c$, and p is a prime or prime power by means of rotating factorial designs. Recently, Georgiou

(2009) constructed orthogonal LHDs from generalized orthogonal designs (ODs), Bingham, Sitter, and Tang (2009) and Lin, Mukerjee, and Tang (2009) presented methods to construct orthogonal and nearly orthogonal LHDs, and Sun, Liu, and Lin (2009, 2010) proposed approaches to constructing orthogonal LHDs so that:

(a) the elementwise square of each column is orthogonal to all columns in the design;

(b) the elementwise product of any two distinct columns is orthogonal to all columns in the design.

As discussed in Bingham, Sitter, and Tang (2009), polynomial model and Gaussian-process model are two kinds of popular models for computer experiments. Orthogonal and nearly orthogonal LHDs are directly useful when polynomial models are considered. Note that the orthogonal LHDs constructed by Ye (1998), Cioppa and Lucas (2007), and Georgiou (2009) also possess the above properties. The rationale for constructing such LHDs has been discussed by Ye (1998) and Sun, Liu, and Lin (2010), among others. In particular, such an orthogonal LHD can guarantee that the estimates of linear effects of all factors are uncorrelated not only with each other, but also with the estimates of all quadratic effects and bilinear interactions when fitting a second-order model. However, if one insists on using Gaussian-process models, exact or near orthogonality can be viewed as a useful stepping stone to space-filling designs (cf., Bingham, Sitter, and Tang (2009)). One can select good space-filling designs within the class of orthogonal or nearly orthogonal LHDs according to selection criteria.

In this article, we introduce some methods to construct orthogonal and nearly orthogonal LHDs with properties (a) and (b); in particular, the resulting nearly orthogonal LHDs with $2^{r+1} + 2$ runs have the minimum correlation between any two distinct columns. The article is organized as follows. In Section 2, we present the definition of OD, and propose some methods to construct 2^r-order ODs. Section 3 devotes itself to constructing orthogonal and nearly orthogonal LHDs from ODs. Some concluding remarks are given in Section 4.

2. Orthogonal Designs and Their Construction

In this section, the definition of OD is given, and construction methods for 2^r-order ODs are provided and illustrated with an example.

Definition 1. An $n \times n$ matrix D is called an n-order OD, denoted by $OD(n)$, if it satisfies the following:

(i) it has real entries $\pm x_1, \ldots, \pm x_n$;

(ii) each column of $\widetilde{D}$ is a permutation of $\{x_1, \ldots, x_n\}$, where $\widetilde{D}$ is a matrix whose elements are obtained from D by changing $-x_i$ to x_i, for $i = 1, \ldots, n$;

(iii) the inner product of any two distinct columns is zero.

In this paper, we take $x_i = ia + b$ with $a \neq 0$, then $x_1, \ldots, x_n$ are equally spaced. We see some ODs in the following example.

Example 1. Three ODs of orders 2, 4 and 8, respectively, are

$$D_1 = \begin{pmatrix} a+b & 2a+b \\ 2a+b & -a-b \end{pmatrix}, \ D_2 = \begin{pmatrix} a+b & 2a+b & -4a-b & 3a+b \\ 2a+b & -a-b & -3a-b & -4a-b \\ 3a+b & 4a+b & 2a+b & -a-b \\ 4a+b & -3a-b & a+b & 2a+b \end{pmatrix}, \text{ and}$$

$$D_3 = \begin{pmatrix} a+b & 2a+b & -4a-b & 3a+b & -8a-b & 7a+b & -5a-b & -6a-b \\ 2a+b & -a-b & -3a-b & -4a-b & -7a-b & -8a-b & -6a-b & 5a+b \\ 3a+b & 4a+b & 2a+b & -a-b & -6a-b & 5a+b & 7a+b & 8a+b \\ 4a+b & -3a-b & a+b & 2a+b & -5a-b & -6a-b & 8a+b & -7a-b \\ 5a+b & 6a+b & -8a-b & 7a+b & 4a+b & -3a-b & a+b & 2a+b \\ 6a+b & -5a-b & -7a-b & -8a-b & 3a+b & 4a+b & 2a+b & -a-b \\ 7a+b & 8a+b & 6a+b & -5a-b & 2a+b & -a-b & -3a-b & -4a-b \\ 8a+b & -7a-b & 5a+b & 6a+b & a+b & 2a+b & -4a-b & 3a+b \end{pmatrix}.$$

For any integer $r \geq 1$, construction approaches for $OD(2^r)$'s are given below.

Lemma 1. *Let*

$$C_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ D_1 = \begin{pmatrix} a+b & 2a+b \\ 2a+b & -a-b \end{pmatrix} \tag{2.1}$$

and, for any integer $r > 1$,

$$C_r = \begin{pmatrix} C_{r-1} & -C^*_{r-1} \\ C_{r-1} & C^*_{r-1} \end{pmatrix} \text{ and } D_r = \begin{pmatrix} D_{r-1} & -D^*_{r-1} - 2^{r-1}aC^*_{r-1} \\ D_{r-1} + 2^{r-1}aC_{r-1} & D^*_{r-1} \end{pmatrix}, \tag{2.2}$$

where $*$ is an operator satisfying (1) $A^{*T}B^* = A^T B$ *for any two matrices A and B, and* (2) $C_r^T D_r^* - D_r^T C_r^* = 0$ *for $r \geq 1$. Then*

(i) $C_r^T C_r = 2^r I_{2^r}$*, where I_{2^r} is an identity matrix of order 2^r;*

(ii) $C_r^T D_r + D_r^T C_r = c_r I_{2^r}$*, where $c_r = (2^{2r}a + 2^r a + 2^{r+1}b)$; and*

(iii) $D_r^T D_r = d_r I_{2^r}$*, where $d_r = a^2 2^r(2^r+1)(2^{r+1}+1)/6 + (2^{2r}+2^r)ab + 2^r b^2$.*

Proof. (i) This is easily proved, since

$$\begin{aligned} C_r^T C_r &= \begin{pmatrix} 2C_{r-1}^T C_{r-1} & 0 \\ 0 & 2C_{r-1}^{*T} C^*_{r-1} \end{pmatrix} = \begin{pmatrix} 2C_{r-1}^T C_{r-1} & 0 \\ 0 & 2C_{r-1}^T C_{r-1} \end{pmatrix} \\ &= 2I_2 \otimes C_{r-1}^T C_{r-1} = \cdots = 2^{r-1} I_{2^{r-1}} \otimes C_1^T C_1 = 2^r I_{2^r}, \end{aligned}$$

where $A \otimes B$ is the Kronecker product of matrices A and B.
(ii) It is easy to see that this holds for $r = 1$, i.e., $C_1^T D_1 + D_1^T C_1 = (6a + 4b)I_2$. Suppose it holds for r. From (2.2), we have

$$C_{r+1}^T D_{r+1} = \begin{pmatrix} 2C_r^T D_r + 2^r a C_r^T C_r & -2^r a C_r^T C_r^* \\ 2^r a C_r^{*T} C_r & 2C_r^T D_r + 2^r a C_r^T C_r \end{pmatrix}, \text{ and}$$
$$D_{r+1}^T C_{r+1} = \begin{pmatrix} 2D_r^T C_r + 2^r a C_r^T C_r & 2^r a C_r^T C_r^* \\ -2^r a C_r^{*T} C_r & 2D_r^T C_r + 2^r a C_r^T C_r \end{pmatrix}.$$

Then (ii) holds by induction, since

$$C_{r+1}^T D_{r+1} + D_{r+1}^T C_{r+1} = (2(2^{2r} a + 2^r a + 2^{r+1} b) + 2^{2r+1} a) I_{2^{r+1}} = c_{r+1} I_{2^{r+1}}.$$

(iii) It is easy to obtain that $D_1^T D_1 = (5a^2 + 6ab + 2b^2)I_2 = d_1 I_2$. Suppose $D_r^T D_r = d_r I_{2^r}$ for r. Let

$$A = 2D_r^{*T} D_r^* + 2^r a(D_r^{*T} C_r^* + C_r^{*T} D_r^*) + 2^{2r} a^2 C_r^{*T} C_r^*.$$

Then from (i), (ii), the supposed condition, and the properties of the $*$ operator, we have

$$\begin{aligned} A &= 2D_r^T D_r + 2^r a(D_r^T C_r + C_r^T D_r) + 2^{2r} a^2 C_r^T C_r = d_{r+1} I_{2^r}, \text{ and} \\ D_{r+1}^T D_{r+1} &= \begin{pmatrix} D_r^T & D_r^T + 2^r a C_r^T \\ -D_r^{*T} - 2^r a C_r^{*T} & D_r^{*T} \end{pmatrix} \begin{pmatrix} D_r & -D_r^* - 2^r a C_r^* \\ D_r + 2^r a C_r & D_r^* \end{pmatrix} \\ &= \begin{pmatrix} A & 2^r a(C_r^T D_r^* - D_r^T C_r^*) \\ 2^r a(D_r^{*T} C_r - C_r^{*T} D_r) & A \end{pmatrix} \\ &= \begin{pmatrix} d_{r+1} I_{2^r} & 0 \\ 0 & d_{r+1} I_{2^r} \end{pmatrix} = d_{r+1} I_{2^{r+1}}. \end{aligned}$$

Thus the proof is complete.

In Lemma 1, the orthogonality of D_r is assured by the $*$ operator, but we still cannot construct such designs since we do not know how the $*$ operator works on the matrices.

Theorem 1. *For C_r and D_r as in Lemma 1, $r = 1, 2, \ldots$,*

(i) *if the $*$ operator works on any square matrix of even order n by interchanging the ith row with the $(n+1-i)$th row, for $i = 1, \ldots, n$, then D_r is an $OD(2^r)$;*

(ii) *if the $*$ operator works on any square matrix of even order by multiplying the entries in the bottom half of the matrix by -1 and leaving those in the top half unchanged, then D_r is an $OD(2^r)$;*

(iii) *if the $*$ operator works on any square matrix of even order by multiplying the entries in the top half of the matrix by -1 and leaving those in the bottom half unchanged, then D_r is an $OD(2^r)$.*

Proof. Consider (i). It is easy to see that D_r satisfies Definition 1 (i) and (ii), so we need to verify the orthogonality of D_r. Based on Lemma 1, we need only check that the $*$ operator in (i) satisfies the conditions (1) and (2) in Lemma 1. Obviously, (1) is satisfied. Now $C_r^T D_r^* - D_r^T C_r^* = 0$ is true for $r = 1$. Suppose it is true for r. From Lemma 1, we have

$$\begin{aligned} C_{r+1}^T D_{r+1}^* &= \begin{pmatrix} C_r^T & C_r^T \\ -C_r^{*T} & C_r^{*T} \end{pmatrix} \begin{pmatrix} D_r^* + 2^r a C_r^* & D_r \\ D_r^* & -D_r - 2^r a C_r \end{pmatrix} \\ &= \begin{pmatrix} 2C_r^T D_r^* + 2^r a C_r^T C_r^* & -2^r a C_r^T C_r \\ -2^r a C_r^{*T} C_r^* & -2C_r^{*T} D_r - 2^r a C_r^{*T} C_r \end{pmatrix}, \text{ and} \\ D_{r+1}^T C_{r+1}^* &= \begin{pmatrix} D_r^T & D_r^T + 2^r a C_r^T \\ -D_r^{*T} - 2^r a C_r^{*T} & D_r^{*T} \end{pmatrix} \begin{pmatrix} C_r^* & C_r \\ C_r^* & -C_r \end{pmatrix} \\ &= \begin{pmatrix} 2D_r^T C_r^* + 2^r a C_r^T C_r^* & -2^r a C_r^T C_r \\ -2^r a C_r^{*T} C_r^* & -2D_r^{*T} C_r - 2^r a C_r^{*T} C_r \end{pmatrix}. \end{aligned}$$

Then from the supposed condition we have $C_{r+1}^T D_{r+1}^* = D_{r+1}^T C_{r+1}^*$, and thus (2) is satisfied by induction. Hence, (i) is true. The conclusions in (ii) and (iii) can be proved similarly. Thus we complete the proof.

Note that the ODs in Example 1 are obtained through Theorem 1 (i). Furthermore, we have the following corollary whose proof is similar to those of Lemma 1 and Theorem 1, and thus is omitted here.

Corollary 1. *If in Lemma* 1, *we let*

$$C_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} a+b & -2a-b \\ 2a+b & a+b \end{pmatrix}, \tag{2.3}$$

and/or for any integer $r > 1$, define C_r and D_r as

$$C_r = \begin{pmatrix} C_{r-1} & C_{r-1}^* \\ C_{r-1} & -C_{r-1}^* \end{pmatrix} \text{ and } D_r = \begin{pmatrix} D_{r-1} & D_{r-1}^* + 2^{r-1} a C_{r-1}^* \\ D_{r-1} + 2^r a C_{r-1} & -D_{r-1}^* \end{pmatrix}, \tag{2.4}$$

then the conclusions (i), (ii), *and* (iii) *in Lemma 1 still hold, and D_r in Theorem 1 is still an $OD(2^r)$.*

Remark 1. The construction here is a generalization of the method used in Sun, Liu, and Lin (2009). In particular, the $OD(2^r)$ constructed in Theorem 1 (iii) with $a = 1$ and $b = 0$ corresponds to the T_r defined in Sun, Liu, and Lin (2009).

From Theorem 1 or Corollary 1, a class of 2^r-order ODs can be conveniently constructed. From Definition 1, it is easy to see that for any resulting $OD(2^r)$ denoted by D, $\widetilde{D}$ is an LHD. From those ODs with some given entries, many orthogonal and nearly orthogonal LHDs can be constructed, as is discussed in the subsequent section.

3. Construction of Orthogonal and Nearly Orthogonal LHDs

In this section, we introduce approaches for constructing orthogonal and nearly orthogonal LHDs from ODs, and investigate their desirable properties.

3.1. Construction of orthogonal LHDs

Here, first, is a result that was proved and used for the construction of LHDs in Ye (1998), Georgiou (2009), and Sun, Liu, and Lin (2009, 2010).

Lemma 2. *Let $L = (D^T, -D^T)^T$ (or $L = (D^T, 0_m, -D^T)^T$) be an $L(2n, m)$ (or $L(2n+1, m)$), where 0_m is an $m \times 1$ column vector with all elements zero. Then L satisfies properties* (a) *and* (b) *unconditionally. Furthermore, if D is a column orthogonal matrix, then L is an orthogonal LHD.*

Based on Definition 1 and Lemmas 1 and 2, we easily have the following.

Theorem 2. *Suppose D is an $OD(n)$ with entries $\pm(ia+b)$, $i = 1, \ldots, n$, where $a \neq 0$. Then*

(i) *for $b = -0.5a$, $L = (D^T, -D^T)^T$ is an orthogonal $L(2n, n)$ with properties* (a) *and* (b)*;*

(ii) *for $b = 0$, $L = (D^T, 0_n, -D^T)^T$ is an orthogonal $L(2n+1, n)$ with properties* (a) *and* (b)*.*

Theorem 3. *Suppose C_r and D_r are as defined in Theorem* 1 *or Corollary 1, r is a positive integer, and let $S_{c2^r\times 2^r} = ((D_r^1)^T, \ldots, (D_r^c)^T)^T$, where $D_r^i = D_r + (i-1)a2^rC_r$. Then*

(i) *for $b = -0.5a$, $L = (S^T_{c2^r\times 2^r}, -S^T_{c2^r\times 2^r})^T$ is an orthogonal $L(c2^{r+1}, 2^r)$ with properties* (a) *and* (b)*;*

(ii) *for $b = 0$, $L = (S^T_{c2^r\times 2^r}, 0_{2^r}, -S^T_{c2^r\times 2^r})^T$ is an orthogonal $L(c2^{r+1}+1, 2^r)$ with properties* (a) *and* (b)*.*

Remark 2. The proposed methods in Theorems 2 and 3 extend those of Sun, Liu, and Lin (2009, 2010). From these methods and the $OD(2^r)$'s constructed in Section 2, many more orthogonal $L(c2^{r+1}, 2^r)$'s and $L(c2^{r+1}+1, 2^r)$'s with properties (a) and (b) can be obtained immediately. In particular, the resulting orthogonal LHDs include those constructed in Sun, Liu, and Lin (2009, 2010) as special cases.

Remark 3. Note that any orthogonal LHD constructed in Theorem 2 has $2n$ or $2n+1$ runs and n factors. With reference to Theorem 3 of Sun, Liu, and Lin (2009), the number of factors in the orthogonal LHD attains the maximum value among all the corresponding orthogonal LHDs satisfying both properties (a) and (b). Furthermore, with reference to Theorem 4 of Sun, Liu, and Lin (2010), the orthogonal LHDs we constructed attain the minimum values of $ave(|t|)$, $t_{\max}$, $ave(|q|)$, and $q_{\max}$ among all the orthogonal LHDs with the same run and factor sizes.

3.2. Construction of nearly orthogonal LHDs

For a design with n runs and m factors, denoted by $D = (d_1, \ldots, d_m)$, where d_i is the ith column of D, let $\rho_{ij} = d_i^T d_j / (d_i^T d_i d_j^T d_j)^{1/2}$. If the mean of the level settings in each d_i for $i = 1, \ldots, m$ is zero, then ρ_{ij} is simply the correlation coefficient between d_i and d_j. With reference to Theorem 2 of Lin et al. (2010), there exist no orthogonal LHDs with more than one factor when the run size is $2^{r+1}+2 = 4\times 2^{r-1}+2$, thus only nearly orthogonal LHDs can be constructed. For this case, LHDs with smaller values of ρ_{ij} are preferred. In fact, from the ODs we can construct nearly orthogonal LHDs with all the ρ_{ij}'s for $i \neq j$ achieving their minimum value.

Theorem 4. *Suppose D is an $OD(2^r)$ for $r > 0$, with $a = 2$ and $b = 1$. Let*

$$L = (D^T, 1_{2^r}, -1_{2^r}, -D^T)^T,$$

where 1_{2^r} is a $2^r \times 1$ column vector with all elements unity. Then L is a nearly orthogonal $L(2^{r+1}+2, 2^r)$ with properties (a), (b), *and*

$$\rho_{ij}(L) = \frac{1}{\sum_{k=0}^{2^r}(2k+1)^2}, \text{ for any } i \neq j.$$

This is the minimum possible value of a correlation coefficient between any two distinct columns.

Proof. It can be easily verified that L is an LHD that satisfies properties (a) and (b). Now consider the value of $\rho_{ij}(L)$ for $i \neq j$. From the definition of L, we have

$$\begin{aligned} L^T L &= (D^T, 1_{2^r}, -1_{2^r}, -D^T)(D^T, 1_{2^r}, -1_{2^r}, -D^T)^T \\ &= 2D^T D + 21_{2^r} 1_{2^r}^T \\ &= 2\sum_{k=1}^{2^r}(2k+1)^2 I_{2^r} + 21_{2^r \times 2^r}, \end{aligned}$$

where $1_{2^r\times 2^r}$ is a $2^r \times 2^r$ matrix with all entries unity. Obviously,

$$\rho_{ij}(L) = \frac{2}{2\sum_{k=0}^{2^r}(2k+1)^2} = \frac{1}{\sum_{k=0}^{2^r}(2k+1)^2}, \text{ for any } i \neq j.$$

Since for any $L(2^{r+1}+2, 2^r)$ $L_0 = (l_1, \ldots, l_{2^r})$ with entries $\pm(2k+1)$ for $k = 0, \ldots, 2^r$, $l_i^T l_i = 2\sum_{k=0}^{2^r}(2k+1)^2$ for $i = 1, \ldots, 2^r$, we need only show $|l_i^T l_j| \geq 2$ for any $i \neq j$. Note that l_i and l_j are permutations of $\{\pm(2k+1), k = 0, \ldots, 2^r\}$, and $\sum_{k=1}^{2^{r+1}+2} l_{ki} = \sum_{k=1}^{2^{r+1}+2} l_{kj} = 0$. Without loss of generality, we take $l_i = (1, 3, \ldots, 2^{r+1}+1, -1, -3, \ldots, -2^{r+1}-1)^T$, i.e., $l_{ki} = -l_{(k+2^r+1)i} = 2k-1$. Then we have

$$\begin{aligned} l_i^T l_j &= \sum_{k=1}^{2^r+1} \left[(2k-1)l_{kj} - (2k-1)l_{(k+2^r+1)j}\right] \\ &= \sum_{k=1}^{2^r+1} \left[(2k-2)l_{kj} - 2kl_{(k+2^r+1)j} + l_{kj} + l_{(k+2^r+1)j}\right] \\ &= 2\sum_{k=1}^{2^r+1} \left[(k-1)l_{kj} - kl_{(k+2^r+1)j}\right]. \end{aligned}$$

Note that both l_{kj} and $l_{(k+2^r+1)j}$ are odd, $k = 1, \ldots, 2^r+1$. The quantity $(k-1)l_{kj} - kd_{(k+2^r+1)j}$ must be odd as $(k-1)l_{kj}$ and $kl_{(k+2^r+1)j}$ cannot be both even or both odd. In addition, 2^r+1 is odd. Thus $\sum_{k=1}^{2^r+1}[(k-1)l_{kj} - kl_{(k+2^r+1)j}]$ gives an odd integer since it is the addition of an odd number of odd integers, and then $|l_i^T l_j| \geq 2$ for $i \neq j$, which means that $\rho_{ij}(L)$ for $i \neq j$ is the minimum possible value. This completes the proof.

The following theorem provides a construction of nearly orthogonal $L(2^{r+1}+3, 2^r)$'s. The proof is similar to that of Theorem 4, and is omitted here.

Theorem 5. *Suppose D is an $OD(2^r)$ for $r > 0$, with $a = 1$ and $b = 1$. Let*

$$L = (D^T,\ 1_{2^r},\ 0_{2^r},\ -1_{2^r},\ -D^T)^T.$$

Then L is a nearly orthogonal $L(2^{r+1}+3, 2^r)$ with properties (a), (b), *and*

$$\rho_{ij}(L) = \frac{1}{\sum_{k=0}^{2^r}(k+1)^2}, \text{ for any } i \neq j.$$

Example 2. From the D_3 given in Example 1, we get the following two nearly

Table 1. ρ_{ij} for the resulting $L(n, m)$, with $n = 2^{r+1} + 2$ or $2^{r+1} + 3$ and $m = 2^r$ for $r < 8$

r	1	1	2	2	3	3	4	4	5	5	6	6	7	7
n	6	7	10	11	18	19	34	35	66	67	130	131	258	259
m	2	2	4	4	8	8	16	16	32	32	64	64	128	128
ρ_{ij}	0.029	0.071	0.006	0.018	0.001	0.003	2E−4	6E−4	3E−5	8E−5	3E−6	1E−5	3E−7	1E−6

orthogonal $L(18, 8)$ and $L(19, 8)$ with $\rho_{ij} \approx 0.001$ and 0.003, respectively:

$$\begin{pmatrix} 3 & 5 & -9 & 7 & -17 & 15 & -11 & -13 \\ 5 & -3 & -7 & -9 & -15 & -17 & -13 & 11 \\ 7 & 9 & 5 & -3 & -13 & 11 & 15 & 17 \\ 9 & -7 & 3 & 5 & -11 & -13 & 17 & -15 \\ 11 & 13 & -17 & 15 & 9 & -7 & 3 & 5 \\ 13 & -11 & -15 & -17 & 7 & 9 & 5 & -3 \\ 15 & 17 & 13 & -11 & 5 & -3 & -7 & -9 \\ 17 & -15 & 11 & 13 & 3 & 5 & -9 & 7 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -5 & 9 & -7 & 17 & -15 & 11 & 13 \\ -5 & 3 & 7 & 9 & 15 & 17 & 13 & -11 \\ -7 & -9 & -5 & 3 & 13 & -11 & -15 & -17 \\ -9 & 7 & -3 & -5 & 11 & 13 & -17 & 15 \\ -11 & -13 & 17 & -15 & -9 & 7 & -3 & -5 \\ -13 & 11 & 15 & 17 & -7 & -9 & -5 & 3 \\ -15 & -17 & -13 & 11 & -5 & 3 & 7 & 9 \\ -17 & 15 & -11 & -13 & -3 & -5 & 9 & -7 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 3 & -5 & 4 & -9 & 8 & -6 & -7 \\ 3 & -2 & -4 & -5 & -8 & -9 & -7 & 6 \\ 4 & 5 & 3 & -2 & -7 & 6 & 8 & 9 \\ 5 & -4 & 2 & 3 & -6 & -7 & 9 & -8 \\ 6 & 7 & -9 & 8 & 5 & -4 & 2 & 3 \\ 7 & -6 & -8 & -9 & 4 & 5 & 3 & -2 \\ 8 & 9 & 7 & -6 & 3 & -2 & -4 & -5 \\ 9 & -8 & 6 & 7 & 2 & 3 & -5 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -2 & -3 & 5 & -4 & 9 & -8 & 6 & 7 \\ -3 & 2 & 4 & 5 & 8 & 9 & 7 & -6 \\ -4 & -5 & -3 & 2 & 7 & -6 & -8 & -9 \\ -5 & 4 & -2 & -3 & 6 & 7 & -9 & 8 \\ -6 & -7 & 9 & -8 & -5 & 4 & -2 & -3 \\ -7 & 6 & 8 & 9 & -4 & -5 & -3 & 2 \\ -8 & -9 & -7 & 6 & -3 & 2 & 4 & 5 \\ -9 & 8 & -6 & -7 & -2 & -3 & 5 & -4 \end{pmatrix}.$$

From the methods proposed in Theorems 4 and 5 and the ODs constructed in Section 2, many nearly orthogonal LHDs with properties (a) and (b) can be constructed directly; these LHDs have a small correlation between any two distinct columns, in particular, the nearly orthogonal $L(2^{r+1} + 2, 2^r)$'s have the minimum correlation. From the two expressions of $\rho_{ij}(L)$ in these theorems, it can be easily checked that the values of $\rho_{ij}(L)$ for $i \neq j$ decrease dramatically as r increases, see Table 1.

4. Concluding Remarks

We have introduced some methods for constructing 2^r-order ODs, and methods for constructing orthogonal and nearly orthogonal LHDs from these ODs. The resulting orthogonal LHDs have the appealing properties (a) and (b), and those with 2^{r+1} or $2^{r+1} + 1$ runs and 2^r factors attain the maximum number of factors (cf., Sun, Liu, and Lin (2009)). The orthogonal LHDs obtained in Sun, Liu, and Lin (2009, 2010) have the same properties, but we can construct more orthogonal LHDs with the same run and factor sizes. Further, we can obtain nearly orthogonal LHDs with $2^{r+1}+2$ or $2^{r+1}+3$ runs and 2^r factors with properties (a) and (b). From Theorem 2 of Lin et al. (2010), we know that orthogonal LHDs with $2^{r+1}+2$ runs do not exist, and we have shown that the newly constructed nearly orthogonal LHDs with $2^{r+1}+2$ runs have the minimum correlation between any two distinct columns.

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References

Bingham, D., Sitter, R. R. and Tang, B. (2009). Orthogonal and nearly orthogonal designs for computer experiments. *Biometrika* **96**, 51-65.

Cioppa, T. M. and Lucas, T. W. (2007). Efficient nearly orthogonal and space-filling Latin hypercubes. *Technometrics* **49**, 45-55.

Georgiou, S. D. (2009). Orthogonal Latin hypercube designs from generalized orthogonal designs. *J. Statist. Plann. Inference* **139**, 1530-1540.

Lin, C. D., Bingham, D., Sitter, R. R. and Tang, B. (2010). A new and flexible method for constructing designs for computer experiments. *Ann. Statist.* **38**, 1460-1477.

Lin, C. D., Mukerjee, R. and Tang, B. (2009). Construction of orthogonal and nearly and orthogonal Latin hypercube designs. *Biometrika* **96**, 243-247.

McKay, M. D., Beckman, R. J. and Conover, W. J. (1979). A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics* **21**, 239-245

Pang, F., Liu, M. Q. and Lin, D. K. J. (2009). A construction method for orthogonal Latin hypercube designs with prime power levels. *Statist. Sinica* **19**, 1721-1728.

Steinberg, D. M. and Lin, D. K. J. (2006). A construction method for orthogonal Latin hypercube designs. *Biometrika* **93**, 279-288.

Sun, F. S., Liu, M. Q. and Lin, D. K. J. (2009). Construction of orthogonal Latin hypercube designs. *Biometrika* **96**, 971-974.

Sun, F. S., Liu, M. Q. and Lin, D. K. J. (2010). Construction of orthogonal Latin hypercube designs with flexible run sizes. *J. Statist. Plann. Inference* **140**, 3236-3242.

Ye, K. Q. (1998). Orthogonal column Latin hypercubes and their application in computer experiments. *J. Amer. Statist. Assoc.* **93**, 1430-1439.

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