International Mathematics Research Notices Advance Access published August 30, 2012
W. Y. C. Chen *et al.* (2012) "Brändén's Conjectures on the Boros–Moll Polynomials,"
International Mathematics Research Notices, rns193, 10 pages.
doi:10.1093/imrn/rns193

Brändén's Conjectures on the Boros–Moll Polynomials

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We prove two conjectures of Brändén on the real-rootedness of the polynomials $Q_n(x)$ and $R_n(x)$ which are related to the Boros–Moll polynomials $P_n(x)$. In fact, we show that both $Q_n(x)$ and $R_n(x)$ form Sturm sequences. The first conjecture implies the 2-log-concavity of $P_n(x)$, and the second conjecture implies the 3-log-concavity of $P_n(x)$.

1 Introduction

In this paper, we prove two conjectures of Brändén [4] concerning the Boros–Moll polynomials. Brändén introduced two polynomials based on the coefficients of the Boros–Moll polynomials and conjectured that these polynomials have only real roots. As pointed out by Brändén, the first conjecture implies the 2-fold log-concavity, or 2-log-concavity, for short, of the Boros–Moll polynomials, whereas the second conjecture implies the 3-log-concavity.

Let us start with some definitions. Given a finite nonnegative sequence $\{a_i\}_{i=0}^n$, we say that it is unimodal if there exists an integer $m \geq 0$ such that

$$a_0 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n,$$

Received May 16, 2012; Accepted July 19, 2012
Communicated by Prof. Percy Deift

and we say that it is log-concave if

$$a_i^2 - a_{i+1}a_{i-1} \geq 0$$

for $1 \leq i \leq n-1$. Define $\mathcal{L}$ to be an operator acting on the sequence $\{a_i\}_{i=0}^n$ as given by

$$\mathcal{L}(\{a_i\}_{i=0}^n) = \{b_i\}_{i=0}^n,$$

where $b_i = a_i^2 - a_{i+1}a_{i-1}$ for $0 \leq i \leq n$ under the convention that $a_{-1} = 0$ and $a_{n+1} = 0$. Clearly, the sequence $\{a_i\}_{i=0}^n$ is log-concave if and only if the sequence $\{b_i\}_{i=0}^n$ is nonnegative. Given a sequence $\{a_i\}_{i=0}^n$, we say that it is k-fold log-concave, or k-log-concave, if $\mathcal{L}^j(\{a_i\}_{i=0}^n)$ is a nonnegative sequence for any $1 \leq j \leq k$. A sequence $\{a_i\}_{i=0}^n$ is said to be infinitely log-concave if it is k-log-concave for all $k \geq 1$. Given a polynomial

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

we say that $f(x)$ is log-concave (or k-log-concave, or infinitely log-concave) if the sequence $\{a_i\}_{i=0}^n$ is log-concave (resp., k-log-concave, infinitely log-concave). Throughout this paper, we shall be concerned with polynomials with real coefficients.

The notion of infinite log-concavity was introduced by Boros and Moll [3] in their study of the following quartic integral:

$$\int_0^\infty \frac{1}{(t^4 + 2xt^2 + 1)^{n+1}}\,\mathrm{d}t.$$

For any $x > -1$ and any nonnegative integer n, they obtained the following formula:

$$\int_0^\infty \frac{1}{(t^4 + 2xt^2 + 1)^{n+1}}\,\mathrm{d}t = \frac{\pi}{2^{n+3/2}(x+1)^{n+1/2}} P_n(x),$$

where

$$P_n(x) = \sum_{j,k} \binom{2n+1}{2j} \binom{n-j}{k} \binom{2k+2j}{k+j} \frac{(x+1)^j (x-1)^k}{2^{3(k+j)}}$$

are the Boros–Moll polynomials. Using Ramanujan's Master Theorem, they derived an alternative expression of $P_n(x)$,

$$P_n(x) = 2^{-2n} \sum_j 2^j \binom{2n-2j}{n-j} \binom{n+j}{j} (x+1)^j. \tag{1.1}$$

For other proofs of (1.1), see Amdeberhan and Moll [1]. Write

$$P_n(x) = \sum_{i=0}^{n} d_i(n) x^i. \tag{1.2}$$

We call $\{d_i(n)\}_{i=0}^n$ a Boros–Moll sequence.

The log-concavity of $\{d_i(n)\}_{i=0}^n$ was conjectured by Moll [17], and it was proved by Kauers and Paule [13] by establishing the following recurrence relations of $d_i(n)$:

$$d_i(n+1) = \frac{n+i}{n+1} d_{i-1}(n) + \frac{4n+2i+3}{2(n+1)} d_i(n), \quad 0 \le i \le n+1, \tag{1.3}$$

$$\begin{aligned} d_i(n+1) = {} & \frac{(4n-2i+3)(n+i+1)}{2(n+1)(n+1-i)} d_i(n) \\ & - \frac{i(i+1)}{(n+1)(n+1-i)} d_{i+1}(n), \quad 0 \le i \le n, \end{aligned} \tag{1.4}$$

$$\begin{aligned} d_i(n+2) = {} & \frac{8n^2+24n+19-4i^2}{2(n+2-i)(n+2)} d_i(n+1) \\ & - \frac{(n+i+1)(4n+3)(4n+5)}{4(n+2-i)(n+1)(n+2)} d_i(n), \quad 0 \le i \le n+1, \end{aligned} \tag{1.5}$$

$$\begin{aligned} d_{i-2}(n) = {} & \frac{(i-1)(2n+1)}{(n+2-i)(n+i-1)} d_{i-1}(n) \\ & - \frac{i(i-1)}{(n+2-i)(n+i-1)} d_i(n), \quad 0 \le i \le n. \end{aligned} \tag{1.6}$$

In fact, (1.5) and (1.6) can be derived from (1.3) and (1.4). Note that Moll [18] independently derived the relation (1.5) and (1.6) via the WZ-method.

Chen and Xia [7] showed that the polynomials $P_n(x)$ are ratio monotone. A sequence of positive real numbers $\{a_i\}_{0 \le i \le n}$ is said to be ratio monotone if

$$\frac{a_n}{a_0} \le \frac{a_{n-1}}{a_1} \le \cdots \le \frac{a_{n-i}}{a_i} \le \cdots \le \frac{a_{n-[\frac{n-1}{2}]}}{a_{[\frac{n-1}{2}]}} \le 1 \tag{1.7}$$

and

$$\frac{a_0}{a_{n-1}} \le \frac{a_1}{a_{n-2}} \le \cdots \le \frac{a_{i-1}}{a_{n-i}} \le \cdots \le \frac{a_{[\frac{n}{2}]-1}}{a_{n-[\frac{n}{2}]}} \le 1. \tag{1.8}$$

Note that for a positive sequence, the ratio monotone property implies both log-concavity and the spiral property. It is worth mentioning that there are approaches to proving log-concavity without using recurrence relations. Llamas and Martínez-Bernal [15] proved that if $f(x)$ is a polynomial with nondecreasing and nonnegative coefficients, then $f(x+1)$ is log-concave. Furthermore, Chen et al. [9] proved that if $f(x)$ is a polynomial with nondecreasing and nonnegative coefficients, then $f(x+1)$ is ratio monotone. From (1.1), it is easily seen that the coefficients of $P_n(x-1)$ are nondecreasing and nonnegative. Hence, $P_n(x)$ are log-concave and ratio monotone. A combinatorial interpretation of the log-concavity of $P_n(x)$ has been found by Chen et al. [6].

Boros and Moll [3] also proposed the following conjecture.

Conjecture 1.1. The sequence $\{d_i(n)\}_{i=0}^n$ is infinitely log-concave. □

The infinite log-concavity of the Boros–Moll polynomials seems to be a difficult problem. As remarked by Kauers and Paule [13], it seems that there is little hope to prove the 2-log-concavity of $\{d_i(n)\}_{i=0}^n$ using recurrence relations. By constructing an intermediate function, Chen and Xia [8] proved the 2-log-concavity of $P_n(x)$ by applying recurrence relations. Based on a technique of McNamara and Sagan [16], Kauers verified the infinite log-concavity of $P_n(x)$ for $n \le 129$.

Brändén [4] presented an approach to Conjecture 1.1 by relating higher-order log-concavity to real-rooted polynomials. Boros and Moll [3] conjectured that for any nonnegative integer n the sequence $\{\binom{n}{k}\}_{k=0}^n$ is infinitely log-concave. Fisk [12], McNamara and Sagan [16], and Stanley independently made the following conjecture which implies the conjecture of Boros and Moll. This conjecture has been proved by Brändén [4].

Theorem 1.2. If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a real-rooted polynomial with nonnegative coefficients, the polynomial

$$a_0^2 + (a_1^2 - a_0 a_2)x + \cdots + (a_{n-1}^2 - a_{n-2} a_n)x^{n-1} + a_n^2 x^n$$

is also real-rooted. □

Brändén's proof is based on a symmetric function identity and the Grace–Walsh–Szegö theorem concerning the location of zeros of multi-affine and symmetric polynomials. Moreover, Brändén obtained a general result on the characterization of nonlinear transformations preserving real-rootedness, in the spirit of the characterization of linear transformations preserving stability given by Borcea and Brändén [2]. From the viewpoint of total positivity, Cardon and Nielsen [5] proposed a conjecture that implies Theorem 1.2. Although the Boros–Moll polynomials $P_n(x)$ are not real-rooted, Brändén [4] introduced two polynomials related to $P_n(x)$, and conjectured that they are real-rooted.

Conjecture 1.3 ([4, Conjecture 8.5]). For any $n \geq 1$, the polynomial

$$Q_n(x) = \sum_{i=0}^{n} \frac{d_i(n)}{i!} x^i \tag{1.9}$$

has only real zeros. □

Conjecture 1.4 ([4, Conjecture 8.6]). For any $n \geq 1$, the polynomial

$$R_n(x) = \sum_{i=0}^{n} \frac{d_i(n)}{(i+2)!} x^i \tag{1.10}$$

has only real zeros. □

As pointed out by Brändén [4], by two theorems of Craven and Csordas on iterated Turán inequalities obtained in [10], the real-rootedness of $Q_n(x)$ implies the 2-log-concavity of $P_n(x)$, and the real-rootedness of $R_n(x)$ implies the 3-log-concavity of $P_n(x)$. Brändén's approach suggests that it might be possible to prove the k-log-concavity of $P_n(x)$ for $k \geq 4$ by using the higher iterated Turán inequalities for real entire functions in the Laguerre–Pólya class. However, little is known about the kth iterated Turán inequalities when $k \geq 4$. It is worth mentioning that Csordas [11] proved the real-rootedness of some polynomials related to $Q_n(x)$.

In this paper, we shall prove the above conjectures by showing that the polynomials $Q_n(x)$ and $R_n(x)$ form Sturm sequences. We say that a polynomial is standard if it is zero or its leading coefficient is positive. Let RZ denote the set of polynomials with only real zeros. Suppose that $f(x) \in \mathrm{RZ}$ is a polynomial of degree n with zeros $\{r_k\}_{k=1}^{n}$, and $g(x) \in \mathrm{RZ}$ is a polynomial of degree m with zeros $\{s_k\}_{k=1}^{m}$. We say that $g(x)$ interlaces $f(x)$

if $n = m+1$ and

$$r_n \le s_{n-1} \le r_{n-1} \le \cdots \le r_2 \le s_1 \le r_1,$$

and we say that $g(x)$ strictly interlaces $f(x)$ if, in addition, they have no common zeros. We use $g(x) \preceq f(x)$ to denote that $g(x)$ interlaces $f(x)$, and use $g(x) \prec f(x)$ to denote that $g(x)$ strictly interlaces $f(x)$. For any real numbers a, b, and c, we assume that $a \in \mathrm{RZ}$ and $a \prec bx + c$. A sequence $\{f_n(x)\}_{n\ge 0}$ of standard polynomials is said to be a Sturm sequence if, for $n \ge 0$, we have $\deg f_n(x) = n$ and

$$f_n(x) \in \mathrm{RZ} \quad \text{and} \quad f_n(x) \prec f_{n+1}(x).$$

To prove that $Q_n(x)$ and $R_n(x)$ are Sturm sequences, we shall use the following sufficient condition, due to Liu and Wang [14], for a polynomial sequence $\{f_n(x)\}_{n\ge 0}$ to form an interlacing sequence.

Theorem 1.5 ([14, Corollary 2.4]). Let $\{f_n(x)\}_{n\ge 0}$ be a sequence of polynomials with nonnegative coefficients and $\deg f_n(x) = n$, which satisfy the following recurrence relation:

$$f_{n+1}(x) = a_n(x) f_n(x) + b_n(x) f_n'(x) + c_n(x) f_{n-1}(x), \tag{1.11}$$

where $a_n(x), b_n(x),$ and $c_n(x)$ are some polynomials with real coefficients. Assume that, for some $n \ge 1$, the following conditions hold:

(i) $f_{n-1}(x), f_n(x) \in \mathrm{RZ}$ and $f_{n-1}(x) \prec f_n(x)$; and
(ii) for any $x \le 0$ both of $b_n(x)$ and $c_n(x)$ are nonpositive, and at least one of them is nonzero.

Then we have $f_{n+1}(x) \in \mathrm{RZ}$ and $f_n(x) \prec f_{n+1}(x)$. □

2 Proofs of Brändén's Conjectures

We first derive recurrence relations for $Q_n(x)$ and $R_n(x)$ based on the recurrence relations (1.3) and (1.5) of the coefficients $d_i(n)$ of the Boros–Moll polynomials $P_n(x)$.

Theorem 2.1. For $n \geq 1$, we have the following recurrence relation:

$$Q_{n+1}(x) = \left(\frac{(2n+1)x}{(n+1)^2} + \frac{8n^2+8n+3}{2(n+1)^2} \right) Q_n(x) - \frac{(4n-1)(4n+1)}{4(n+1)^2} Q_{n-1}(x) + \frac{x}{(n+1)^2} Q'_n(x). \tag{2.1}$$

□

Proof. For $n \geq 1$, relation (2.1) can be rewritten as

$$4(n+1)^2 d_i(n+1) = 2(8n^2+8n+3+2i)d_i(n) + 4i(2n+1)d_{i-1}(n) - (16n^2-1)d_i(n-1), \tag{2.2}$$

where $0 \leq i \leq n+1$. From (1.3) it follows that

$$d_{i-1}(n) = \frac{n+1}{n+i} d_i(n+1) - \frac{4n+2i+3}{2(n+i)} d_i(n). \tag{2.3}$$

Substituting (2.3) into (2.2), we obtain

$$d_i(n+1) = \frac{8n^2+8n+3-4i^2}{2(n+1-i)(n+1)} d_i(n) - \frac{(n+i)(4n-1)(4n+1)}{4n(n+1)(n+1-i)} d_i(n-1). \tag{2.4}$$

It is easily checked that the above relation (2.4) coincides with (1.5) with n replaced by $n-1$. This completes the proof. ■

Using the above recurrence relation and the criterion of Liu and Wang, we can deduce that the polynomials $Q_n(x)$ form a Sturm sequence. This leads to an affirmative answer to Conjecture 1.3.

Theorem 2.2. The polynomial sequence $\{Q_n(x)\}_{n\geq 0}$ is a Sturm sequence. □

Proof. Clearly, we have $\deg(Q_n(x)) = n$. It suffices to prove that $Q_n(x) \in \mathrm{RZ}$ and $Q_n(x) \prec Q_{n+1}(x)$ for any $n \geq 0$. We use induction on n. By convention,

$$Q_0(x), Q_1(x) \in \mathrm{RZ} \quad \text{and} \quad Q_0(x) \prec Q_1(x).$$

Assume that

$$Q_{n-1}(x), \quad Q_n(x) \in \mathrm{RZ} \quad \text{and} \quad Q_{n-1}(x) \prec Q_n(x).$$

We proceed to verify that

$$Q_{n+1}(x) \in \mathrm{RZ} \quad \text{and} \quad Q_n(x) \prec Q_{n+1}(x).$$

We see that the recurrence relation (2.1) of $Q_n(x)$ is of the form (1.11) in Theorem 1.5, where the polynomials $a_n(x)$, $b_n(x)$, and $c_n(x)$ are given by

$$\begin{aligned}
a_n(x) &= \frac{(2n+1)x}{(n+1)^2} + \frac{8n^2+8n+3}{2(n+1)^2}, \\
b_n(x) &= \frac{x}{(n+1)^2}, \\
c_n(x) &= -\frac{(4n-1)(4n+1)}{4(n+1)^2}.
\end{aligned}$$

For $n \geq 1$ and $x \leq 0$, one can check that

$$b_n(x) \leq 0 \quad \text{and} \quad c_n(x) < 0.$$

In view of Theorem 1.5, we find that $Q_{n+1}(x) \in \mathrm{RZ}$ and $Q_n(x) \prec Q_{n+1}(x)$. This completes the proof. ■

The following recurrence relation for $R_n(x)$ can be proved in a way similar to the proof of Theorem 2.1.

Theorem 2.3. For $n \geq 1$, we have

$$\begin{aligned}
R_{n+1}(x) = &\left(\frac{(2n+1)x}{(n+1)(n+3)} + \frac{8n^2+8n+7}{2(n+1)(n+3)} \right) R_n(x) \\
&- \frac{(4n-1)(4n+1)(n-2)}{4n(n+1)(n+3)} R_{n-1}(x) + \frac{5x}{(n+1)(n+3)} R_n'(x).
\end{aligned} \tag{2.5}$$

□

Using the above recurrence relation, we obtain the following theorem, which leads to an affirmative answer to Conjecture 1.4.

Theorem 2.4. The polynomial sequence $\{R_n(x)\}_{n\geq 0}$ is a Sturm sequence. $\square$

Proof. The proof is analogous to that of Theorem 2.2. It is routine to verify that

$$R_0(x),\, R_1(x),\, R_2(x),\, R_3(x) \in \mathrm{RZ} \quad \text{and} \quad R_0(x) \prec R_1(x) \prec R_2(x) \prec R_3(x).$$

It remains to show that $R_n(x) \in \mathrm{RZ}$ and $R_{n-1}(x) \prec R_n(x)$ for $n \geq 3$. We use induction on n. Assume that

$$R_{n-1}(x), \quad R_n(x) \in \mathrm{RZ}, \quad \text{and} \quad R_{n-1}(x) \prec R_n(x).$$

We wish to prove that

$$R_{n+1}(x) \in \mathrm{RZ} \quad \text{and} \quad R_n(x) \prec R_{n+1}(x).$$

The recurrence relation (2.5) of $R_n(x)$ is of the form (1.11) in Theorem 1.5, and the polynomials $a_n(x)$, $b_n(x)$, and $c_n(x)$ are given by

$$a_n(x) = \frac{(2n+1)x}{(n+1)(n+3)} + \frac{8n^2+8n+7}{2(n+1)(n+3)},$$
$$b_n(x) = \frac{5x}{(n+1)(n+3)},$$
$$c_n(x) = -\frac{(4n-1)(4n+1)(n-2)}{4n(n+1)(n+3)}.$$

For $n \geq 3$ and $x \leq 0$, we find that

$$b_n(x) \leq 0 \quad \text{and} \quad c_n(x) < 0.$$

By Theorem 1.5, we conclude that $R_{n+1}(x) \in \mathrm{RZ}$ and $R_n(x) \prec R_{n+1}(x)$. This completes the proof. $\blacksquare$

Acknowledgements

We wish to thank Petter Brändén and the referees for valuable comments.

Funding

This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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