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On zero-sum subsequences of length $k\exp(G)$

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ABSTRACT

Let G be an additive finite abelian group of exponent $\exp(G)$. For every positive integer k, let $\mathrm{s}_{k\exp(G)}(G)$ denote the smallest integer t such that every sequence over G of length t contains a zero-sum subsequence of length $k\exp(G)$. We prove that if $\exp(G)$ is sufficiently larger than $\frac{|G|}{\exp(G)}$ then $\mathrm{s}_{k\exp(G)}(G) = k\exp(G) + \mathrm{D}(G) - 1$ for all $k \geqslant 2$, where $\mathrm{D}(G)$ is the Davenport constant of G.

1. Introduction

Let G be an additive finite abelian group with exponent $\exp(G) = m$. Let $\mathrm{D}(G)$ denote the Davenport constant of G, which is defined as the smallest integer t such that every sequence S over G of length $|S| \geqslant t$ contains a nonempty zero-sum subsequence. For every positive integer k, let $\mathrm{s}_{km}(G)$ denote the smallest integer t such that every sequence S over G of length $|S| \geqslant t$ contains a zero-sum subsequence of length km. For $k = 1$, we abbreviate $\mathrm{s}_m(G)$ to $\mathrm{s}(G)$ which is called the Erdős–Ginzburg–Ziv constant of G. The invariant $\mathrm{s}(G)$ has been studied by many authors (for example, see [1,2,5,6,8,9,17,16,23,24,27,29,30]). The famous Erdős–Ginzburg–Ziv theorem [7] asserts that

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$\mathsf{s}_{|G|}(G) \leqslant 2|G| - 1$ and the equality holds for cyclic groups. In 1996, the first author [11] proved that

$$\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1$$

provided that $km \geqslant |G|$.

Let T be a zero-sum free sequence over G of length $|T| = \mathsf{D}(G) - 1$ and let

$$S = 0^{km-1}T.$$

Clearly, S contains no zero-sum subsequence of length km. Therefore,

$$\mathsf{s}_{km}(G) \geqslant km + \mathsf{D}(G) - 1 \tag{1.1}$$

holds for every $k \geqslant 1$.

The first author and Thangadurai [15] noticed that if $km < \mathsf{D}(G)$ then $\mathsf{s}_{km}(G) > km + \mathsf{D}(G) - 1$, and introduced the invariant $\ell(G)$ which is defined as the smallest integer t such that $\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1$ holds for every $k \geqslant \ell$. From the above we know that

$$\frac{\mathsf{D}(G)}{m} \leqslant \ell(G) \leqslant \frac{|G|}{m}. \tag{1.2}$$

For cyclic groups G, we clearly have $\ell(G) = 1$ by the Erdős–Ginzburg–Ziv theorem. For finite abelian groups G of rank two we can deduce that $\ell(G) = 2$ from some known results (see Proposition 4.1). For finite abelian p-groups, $\mathsf{s}_{km}(G)$ has been studied in [10,15,25]. For related papers we refer to [4,22,32]. Our main result in this paper is:

Theorem 1.1. *Let H be an arbitrary finite abelian group with $\exp(H) = m \geqslant 2$, and let $G = C_{mn} \oplus H$. If $n \geqslant 2m|H| + 2|H|$, then $\mathsf{s}_{kmn}(G) = kmn + \mathsf{D}(G) - 1$ for all positive integers $k \geqslant 2$, and therefore $\ell(G) = 2$.*

2. Preliminaries

Our notation and terminology are consistent with [13] and [20]. We briefly gather some key notions and fix the notations concerning sequences over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}$, we set $[a, b] = \{x \in \mathbb{N}\colon\ a \leqslant x \leqslant b\}$. Throughout this paper, all abelian groups will be written additively, and for $n, r \in \mathbb{N}$, we denote by C_n the cyclic group of order n, and denote by C_n^r the direct sum of r copies of C_n.

Let G be a finite abelian group and $\exp(G)$ its exponent. A sequence S over G will be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G,$$

and we call

$$|S| = \ell \in \mathbb{N}_0 \quad \text{the } \textit{length} \quad \text{and} \quad \sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathrm{v}_g(S) g \in G \quad \text{the } \textit{sum} \text{ of } S.$$

Let $\mathrm{supp}(S) = \{g \in G\colon \mathrm{v}_g(S) > 0\}$. For every $r \in [1, \ell]$ define

$$\Sigma_r(S) = \{\sigma(T)\colon T \mid S,\ |T| = r\}$$

where $T \mid S$ means T is a subsequence of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$.
- a *short zero-sum sequence* over G if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

For every element $g \in G$, we set $g + S = (g + g_1) \cdot \dots \cdot (g + g_l)$. If $\varphi : G \to H$ is a group homomorphism, then $\varphi(S) = \varphi(g_1) \cdot \dots \cdot \varphi(g_l)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

Let $\eta(G)$ be the smallest integer t such that every sequence S over G of length $|S| \geqslant t$ contains a short zero-sum subsequence.

Lemma 2.1. *(See [19, Theorem 4.2.7].) We have $\eta(G) \leqslant |G|$ and $\mathrm{s}(G) \leqslant |G| + \exp(G) - 1$.*

Lemma 2.2. *Let n, k, t be three positive integers with $2 \leqslant t < \frac{n}{2} + 1$, and let S be a sequence over C_n of length $|S| = (k+1)n - t$. Suppose that S contains no zero-sum subsequence of length kn. Then, there exist two distinct elements $a, b \in C_n$ such that*

$$\mathrm{v}_a(S) + \mathrm{v}_b(S) \geqslant (k+1)n - 2t + 2. \tag{2.1}$$

Furthermore, if $2 \leqslant t < \frac{n+5}{3}$, then the pair of $\{a, b\}$ satisfying inequality (2.1) is uniquely determined by S.

Proof. We can prove the existence of $\{a, b\}$ satisfying (2.1) in a similar way to the proof of [29, Theorem 5] and we omit it here.

Now assume that $1 < t < \frac{n+5}{3}$. Suppose that $\mathrm{v}_c(S) + \mathrm{v}_d(S) \geqslant (k+1)n - 2t + 2$ for another pair $\{c, d\} \neq \{a, b\}$. Since $0 \notin \sum_{kn}(S)$, $\mathrm{v}_g(S) \leqslant kn - 1$ for every $g \in C_n$. It follows that $\mathrm{v}_g(S) \geqslant n - 2t + 3$ for every $g \in \{a, b, c, d\}$. Without loss of generality we assume that $c \notin \{a, b\}$. Therefore, $\mathrm{v}_a(S) + \mathrm{v}_b(S) + \mathrm{v}_c(S) \geqslant (k+1)n - 2t + 2 + (n - 2t + 3) > (k+1)n - t = |S|$, yielding a contradiction. Hence $\{a, b\}$ is the unique pair satisfying (2.1). $\square$

We also need the following easy result which is a straightforward consequence of [19, Lemma 4.2.5] and we omit the proof here.

Lemma 2.3. *Let $m \in \mathbb{N}$, and let H be a finite abelian group with $\exp(H) \mid m$. Let $G = C_{mn} \oplus H$. Then, $\mathrm{D}(G) \leqslant mn + \eta(C_m \oplus H) - m \leqslant mn + m|H| - m$.*

3. Proof of Theorem 1.1

As mentioned in the Introduction, $\mathrm{s}_{kmn}(G) \geqslant kmn + \mathrm{D}(G) - 1$. It suffices to prove that $\mathrm{s}_{kmn}(G) \leqslant kmn + \mathrm{D}(G) - 1$. Let S be any sequence over G of length $|S| = kmn + \mathrm{D}(G) - 1$. We need to show that S contains a zero-sum subsequence of length kmn.

Assume to the contrary that S contains no zero-sum subsequence of length kmn. Let $\varphi : G = C_{mn} \oplus H \to C_m \oplus H$ be the natural homomorphism with $\ker(\varphi) = C_n$ (up to isomorphism).

By applying $\mathrm{s}(\varphi(C_{mn} \oplus H)) = \mathrm{s}(C_m \oplus H)$ on $\varphi(S)$ repeatedly, we can get a decomposition $S = S_1 \cdot \cdots \cdot S_r \cdot S'$ with

$$|S_i| = m, \quad \sigma(S_i) \in \ker(\varphi) \text{ for every } i \in [1, r] \tag{3.1}$$

and $\mathrm{s}(C_m \oplus H) - m \leqslant |S'| \leqslant \mathrm{s}(C_m \oplus H) - 1$. Therefore,

$$r = \left\lceil \frac{|S| - \mathrm{s}(C_m \oplus H) + 1}{m} \right\rceil. \tag{3.2}$$

Let

$$U = \sigma(S_1)\sigma(S_2) \cdot \cdots \cdot \sigma(S_r).$$

It follows from $0 \notin \Sigma_{kmn}(S)$ that $0 \notin \Sigma_{kn}(U)$. Since $\mathrm{D}(G) \geqslant mn$ and $\mathrm{s}(C_m \oplus H) \leqslant m \cdot |H| + m - 1$ by Lemma 2.1, we infer that

$$\begin{aligned} |U| = r &\geqslant \frac{|S| - \mathrm{s}(C_m \oplus H) + 1}{m} \\ &\geqslant \frac{(kmn + \mathrm{D}(G) - 1) - \mathrm{s}(C_m \oplus H) + 1}{m} \\ &\geqslant \frac{kmn + mn - (m \cdot |H| + m - 1)}{m} \\ &= (k+1)n - |H| - \frac{m-1}{m}. \end{aligned}$$

Therefore

$$|U| = r \geqslant (k+1)n - |H| - \frac{m-1}{m}. \tag{3.3}$$

Let

$$t = (k+1)n - r.$$

Since $0 \notin \Sigma_{kn}(U)$, $r = |U| \leqslant (k+1)n-2$ by the Erdős–Ginzburg–Ziv theorem. It follows that $t \geqslant 2$. By (3.3) and the hypothesis that $n \geqslant 2m|H| + 2|H| > 5|H|$, we get

$$t \leqslant \frac{n}{5} < \frac{n+5}{3}.$$

It follows from Lemma 2.2 that there exists a unique pair of $\{a, b\}$ such that

$$\mathrm{v}_a(U) + \mathrm{v}_b(U) \geqslant (k+1)n - 2t + 2.$$

Denote by Ω the set consisting of all decompositions of S satisfying (3.1) and (3.2). Choose a decomposition

$$S = S_1 \cdot S_2 \cdot \cdots \cdot S_r \cdot S' \in \Omega$$

such that $\mathrm{v}_a(U) + \mathrm{v}_b(U)$ attains the minimal value. Let

$$\ell = \mathrm{v}_a(U) + \mathrm{v}_b(U).$$

By renumbering if necessary we assume that $\sigma(S_i) \in \{a, b\}$ for all $i \in [1, \ell]$. Let

$$W = \prod_{i=1}^{\ell} S_i.$$

From $t < \frac{n+5}{3}$ and $n \geqslant 2m|H| + 2|H|$ we derive that

$$\ell \geqslant (k+1)n - 2t + 2 > m.$$

Claim 3.1. *Let W_0 be a subsequence of W of length $|W_0| = m$. If $\sigma(W_0) \in \ker(\varphi)$ then $\sigma(W_0) \in \{a, b\}$.*

Proof. Assume to the contrary that $\sigma(W_0) \notin \{a, b\}$. Since $|W_0| = m$, by renumbering we may assume that $W_0 \mid S_1 \cdot S_2 \cdot \cdots \cdot S_v$ for some $v \in [1, m]$. Then S has a decomposition

$$S = S_{v+1} \cdot S_{v+2} \cdot \cdots \cdot S_r \cdot W_0 \cdot S'_2 \cdot S'_3 \cdot \cdots \cdot S'_v \cdot S'' \in \Omega$$

where $|S'_i| = m$ and $\sigma(S'_i) \in \ker(\varphi)$ for every $i \in [2, v]$.

Let

$$U_1 = \sigma(S_{v+1}) \cdot \sigma(S_{v+2}) \cdot \cdots \cdot \sigma(S_r) \cdot \sigma(W_0) \cdot \sigma\bigl(S'_2\bigr) \cdot \cdots \cdot \sigma\bigl(S'_v\bigr).$$

It follows from $0 \notin \Sigma_{kmn}(S)$ that $0 \notin \Sigma_{kn}(U_1)$. By Lemma 2.2, there is a unique pair of $\{a_1, b_1\}$ such that

$$\mathrm{v}_{a_1}(U_1) + \mathrm{v}_{b_1}(U_1) \geqslant (k+1)n - 2t + 2.$$

Since $0 \notin \Sigma_{kn}(U_1)$, we have $\mathrm{v}_{a_1}(U_1) \leqslant kn-1$ and $\mathrm{v}_{b_1}(U_1) \leqslant kn-1$. It follows that

$$\mathrm{v}_{a_1}(U_1) \geqslant n-2t+3 \quad \text{and} \quad \mathrm{v}_{b_1}(U_1) \geqslant n-2t+3.$$

If $a_1 \notin \{a,b\}$, then $r = |U_1| \geqslant \mathrm{v}_a(U_1)+\mathrm{v}_b(U_1)+\mathrm{v}_{a_1}(U_1) \geqslant \mathrm{v}_a(U)+\mathrm{v}_b(U)-v+\mathrm{v}_{a_1}(U_1) \geqslant (k+1)n-2t+2-v+n-2t+3 \geqslant (k+1)n-t+(n-3t-m+5) > (k+1)n-t = r$, a contradiction. Therefore, $a_1 \in \{a,b\}$. Similarly, $b_1 \in \{a,b\}$. Hence, $\{a_1,b_1\} = \{a,b\}$. But $\mathrm{v}_a(U_1)+\mathrm{v}_b(U_1) < \mathrm{v}_a(U)+\mathrm{v}_b(U)$, a contradiction to the minimality of U. This proves Claim 3.1. □

For every $h \in \varphi(G) = C_m \oplus H$, let W_h be the subsequence of W such that $\varphi(W_h) = h^{\mathrm{v}_h(\varphi(W))}$.

Claim 3.2. *If* $|W_h| \geqslant m+1$ *then* $|\operatorname{supp}(W_h)| \leqslant 2$.

Proof. Assume to the contrary that $|W_h| \geqslant m+1$ and $|\operatorname{supp}(W_h)| \geqslant 3$ for some $h \in \varphi(G) = C_m \oplus H$. Take three distinct elements $g_0, g_1, g_2 \in \operatorname{supp}(W_h)$. Let W' be a subsequence of $W_h(g_0g_1g_2)^{-1}$ of length $|W'| = m-2$. Then, $W'g_0g_1, W'g_0g_2$ and $W'g_1g_2$ are three subsequences of W_h each having sum in $\ker(\varphi) = C_n$. But the sums $\sigma(W'g_0g_1)$, $\sigma(W'g_0g_2)$, $\sigma(W'g_1g_2)$ are pairwise distinct, a contradiction to Claim 3.1. This proves Claim 3.2. □

So, for every $|W_h| \geqslant m+1$ we have

$$W_h = x_h^{u_h} y_h^{v_h},$$

where $x_h, y_h \in G$, $u_h \geqslant v_h \geqslant 0$ and $u_h + v_h = |W_h| = \mathrm{v}_h(\varphi(W))$.

Write

$$u_h = p_h m + r_h \quad \text{and} \quad v_h = q_h m + s_h$$

where $p_h, r_h, q_h, s_h \in \mathbb{N}_0$ and $r_h, s_h \in [0, m-1]$.

For every $h \in \varphi(G) = C_m \oplus H$ with $|W_h| \geqslant m+1$, W_h has the following decomposition

$$W_h = \underbrace{x_h^m \cdot \ldots \cdot x_h^m}_{p_h} \underbrace{y_h^m \cdot \ldots \cdot y_h^m}_{q_h} \left(x_h^{r_h} y_h^{s_h}\right).$$

Let

$$W' = \prod_{h \in C_m \oplus H,\, |W_h| \geqslant m+1} \underbrace{x_h^m \cdot \ldots \cdot x_h^m}_{p_h} \underbrace{y_h^m \cdot \ldots \cdot y_h^m}_{q_h} = T_1 T_2 \cdot \ldots \cdot T_f$$

where $f = \sum_{h \in C_m \oplus H,\, |W_h| \geqslant m+1} (p_h + q_h)$ and for each $i \in [1,f]$ we have $T_i = x_h^m$ or $T_i = y_h^m$ for some $h \in C_m \oplus H$.

Let

$$R=\prod_{i=1}^{f}\sigma(T_i).$$

It follows from Claim 3.1 that $\mathrm{supp}(R)\subseteq\{a,b\}$. Without loss of generality we assume that

$$\mathrm{v}_a(R)\geqslant \mathrm{v}_b(R).$$

Let $\lambda=\mathrm{v}_a(R)$. Then,

$$\begin{aligned}
\lambda=\mathrm{v}_a(R)&\geqslant\frac{|R|}{2}\\
&=\frac{f}{2}=\frac{\sum_{h\in C_m\oplus H,\,|W_h|\geqslant m+1}(p_h+q_h)}{2}\\
&=\frac{\sum_{h\in C_m\oplus H,\,|W_h|\geqslant m+1}(|W_h|-r_h-s_h)}{2m}\\
&=\frac{|W|-\sum_{h\in C_m\oplus H,\,|W_h|\geqslant m+1}(r_h+s_h)-\sum_{h\in C_m\oplus H,\,|W_h|\leqslant m}|W_h|}{2m}\\
&\geqslant\frac{|W|-(2m-2)|C_m\oplus H|}{2m}\geqslant\frac{kn+(n-2m|H|)}{2}.
\end{aligned}$$

So we have

$$\lambda\geqslant\frac{kn+(n-2m|H|)}{2}. \tag{3.4}$$

By renumbering we may assume that

$$\sigma(T_1)=\cdots=\sigma(T_\lambda)=a.$$

Let $T_1=x^m$ and $S'=-x+S$. Then

$$S'=T_1'\cdot\cdots\cdot T_\lambda' S'',$$

where $T_i'=-x+T_i$ for every $i\in[1,\lambda]$, and $T_1'=0^m,\sigma(T_i')=0$ for each $i\in[1,\lambda]$.

By (3.4) and the hypothesis of the theorem we have

$$\left|T_1'\cdot\cdots\cdot T_\lambda'\right|=m\lambda\geqslant \mathrm{D}(G)-1.$$

Therefore,

$$\left|S''\right|=|S|-\left|T_1'\cdot\cdots\cdot T_\lambda'\right|=kmn+\mathrm{D}(G)-1-\left|T_1'\cdot\cdots\cdot T_\lambda'\right|\leqslant kmn.$$

Let S_0 be the maximal (in length) zero-sum subsequence of S''. Then, $|S''| - |S_0| = |S''S_0^{-1}| \leqslant \mathrm{D}(G) - 1$. Hence,

$$|S''| - \mathrm{D}(G) + 1 \leqslant |S_0| \leqslant |S''| \leqslant kmn.$$

Note that $|0^m T_2' \cdot \cdots \cdot T_\lambda' S_0| = |S| - |S''| + |S_0| = kmn + \mathrm{D}(G) - 1 - (|S''| - |S_0|) \geqslant kmn$ and $|S_0| \leqslant kmn$, there exist $m' \in [0, m]$ and $\lambda' \in [0, \lambda]$ such that

$$|0^{m'} T_2' \cdot \cdots \cdot T_{\lambda'}' S_0| = kmn.$$

So, $0^{m'} T_2' \cdot \cdots \cdot T_{\lambda'}' S_0$ is a zero-sum subsequence of length kmn and therefore $x^{m'} T_2 \cdot \cdots \cdot T_{\lambda'}(x + S_0)$ is a zero-sum subsequence of S, a contradiction. This proves that $\mathrm{s}_{kmn}(G) = kmn + \mathrm{D}(G) - 1$ for every $k \geqslant 2$. Now $\ell(G) = 2$ follows from (1.2).

4. Concluding remarks and open problems

In this section we shall give some concluding remarks and some open problems. For finite abelian groups of rank two we have

Proposition 4.1. *Let $G = C_m \oplus C_n$ with $1 < m \mid n$. Then, $\ell(G) = 2$.*

Let G be a finite abelian group and let d be a positive integer. Let $\mathrm{s}_{d\mathbb{N}}(G)$ be the smallest integer t such that every sequence over G of length at least t contains a zero-sum subsequence of length divided by d.

Lemma 4.2. *Let $G = C_m \oplus C_n$ with $1 < m \mid n$. Then,*

(1) $\mathrm{s}(G) = 2n + 2m - 3$ *(see [20, Theorem 5.8.3]),*
(2) $\mathrm{s}_{n\mathbb{N}}(G) = 2n + m - 2$ *(see [21, Theorem 5.2]).*

Proof of Proposition 4.1. For any positive integer $k \geqslant 2$, it suffices to prove that $\mathrm{s}_{kn}(G) \leqslant kn + \mathrm{D}(G) - 1$. Let S be a sequence over G of length $kn + \mathrm{D}(G) - 1 = kn + n + m - 2$. We need to prove that S contains a zero-sum subsequence of length kn.

We proceed by induction on k. For $k = 2$, by Lemma 4.2(1), S contains a zero-sum subsequence S_1 of length n. Since $3n > |SS_1^{-1}| = 2n + m - 2$, by Lemma 4.2(2), SS_1^{-1} contains a zero-sum subsequence S_2 of length $|S_2| \in \{n, 2n\}$. Therefore, either S_1S_2 or S_2 is a zero-sum subsequence of S of length $2n$.

Now suppose that the proposition holds for $k = r$, we want to prove it for $k = r + 1$. By Lemma 4.2(1), S contains a zero-sum subsequence T_1 of length n. Since $|ST_1^{-1}| = (r + 1)n + \mathrm{D}(G) - 1 - n = rn + \mathrm{D}(G) - 1$, by induction hypothesis, ST_1^{-1} contains a zero-sum subsequence T_2 of length $|T_2| = rn$. So, T_1T_2 is a zero-sum subsequence of S of length $|T_1T_2| = (r + 1)n$. $\square$

Let $r \in [1, \mathrm{D}(G) - 1]$, and let S be a sequence over G of length $|S| = |G| + r - 1$ with $0 \notin \Sigma_{|G|}(S)$. In 1999, Bollobás and Leader [3] considered the problem of bounding $|\Sigma_{|G|}(S)|$ from below.

For every $r \in [1, \mathrm{D}(G) - 1]$, define

$$f(G; r) = \max\{|\Sigma(T)|\colon\ |T| = r,\ T \text{ is a zero-sumfree sequence over } G\}.$$

$f(G; r)$ has been studied recently by several authors (for example, see [14,18,26]).

Proposition 4.3. *Let H be an arbitrary finite abelian group with $\exp(H) = m \geqslant 2$, and let $G = C_{mn} \oplus H$. Let $r \in [1, \mathrm{D}(G) - 1]$ and $k \geqslant 3$, and let S be a sequence over G of length $|S| = kmn + r - 1$. Suppose that $n \geqslant 2m|H| + 2|H|$. If $0 \notin \Sigma_{kmn}(S)$ then $|\Sigma_{kmn}(S)| \geqslant f(G; r)$.*

Proof. Similarly to the proof of Theorem 1.1 we can find an element $x \in G$ such that $x + S$ has a factorization

$$x + S = T_1' \cdot \dots \cdot T_\lambda' S''$$

with $T_1' = 0^m$, $\sigma(T_i') = 0$ and $|T_i'| = m$ for each $i \in [1, \lambda]$, and

$$\lambda \geqslant \frac{(k-1)n + (n - 2m|H|)}{2}.$$

By Lemma 2.3, $r \leqslant \mathrm{D}(G) \leqslant mn + m|H| - m$. It follows from $k \geqslant 3$ and $n \geqslant 2m|H| + 2|H|$ that

$$|S''| \leqslant kmn.$$

Let S_0 be the maximal (in length) zero-sum subsequence of S''. Then,

$$|S''| - |S_0| = |S'' S_0^{-1}| \leqslant \mathrm{D}(G) - 1.$$

If $\lambda m + |S_0| = |T_1' \cdot \dots \cdot T_\lambda' S_0| \geqslant kmn$, then similarly to the proof of Theorem 1.1 we can prove that $0 \in \Sigma_{kmn}(x + S) = \Sigma_{kmn}(S)$, a contradiction. Therefore,

$$\lambda m + |S_0| = |T_1' \cdot \dots \cdot T_\lambda' S_0| \leqslant kmn.$$

Hence,

$$|S'' S_0^{-1}| \geqslant r.$$

Let W be an arbitrary subsequence of $S'' S_0^{-1}$ of length $|W| = r$, and let $W' = S'' S_0^{-1} W^{-1}$. Then,

$$x + S = 0^m T_2' \cdot \dots \cdot T_\lambda' S_0 W' W.$$

From the maximality of S_0 we know that $W'W$ is zero-sum free. So, W is a zero-sum free sequence of length r. Hence,

$$\left|\Sigma(W)\right| \geqslant f(G;r).$$

For every $y \in \Sigma(W)$, there is a nonempty subsequence $W_0 \mid W$ such that $y = \sigma(W_0)$. Therefore, $\sigma(W') + y = \sigma(W'W_0) = \sigma(0^m T_2' \cdot \cdots \cdot T_\lambda' S_0 W' W_0)$. Note that $|0^m T_2' \cdot \cdots \cdot T_\lambda' S_0 W' W_0| = |S| - |W W_0^{-1}| \geqslant kmn$, in a similar way to the proof of Theorem 1.1, we can prove that $\sigma(W') + y \in \Sigma_{kmn}(x + S) = \Sigma_{kmn}(S)$. This proves that $|\Sigma_{kmn}(S)| \geqslant |\sigma(W') + \Sigma(W)| = |\Sigma(W)| \geqslant f(G;r)$. □

We end the paper by discussing some conjectures related to the problems we investigated.

Conjecture 4.4. *(See [15].) For every non-cyclic finite abelian group G the sequence*

$$\left\{\mathsf{s}_{km}(G) - km\right\}_{k=1}^{\ell(G)-1}$$

is strictly decreasing.

Conjecture 4.5. *(See [25].) If $G = C_n^r$ then $s_{kn}(G) = kn + r(n-1)$ holds for every positive integer $k \geqslant r$.*

Let G be a finite abelian group with $\exp(G) = m$. For every $k \in \mathbb{N}$, let $\eta_{km}(G)$ denote the smallest integer t such that every sequence S over G of length $|S| \geqslant t$ contains a zero-sum subsequence T of length $|T| \in [1, km]$.

Conjecture 4.6. *Let G be a finite abelian group with $\exp(G) = m$. Then, $\mathsf{s}_{km}(G) = \eta_{km}(G) + km - 1$ for every $k \in \mathbb{N}$.*

For $k = 1$, Conjecture 4.6 was formulated by the first author in [12]. If $km \geqslant \mathsf{D}(G)$, we clearly have that $\eta_{km}(G) = \mathsf{D}(G)$. So, Conjecture 4.6, if true, together with (1.2) would imply the following

Conjecture 4.7. *Let G be a finite abelian group with $\exp(G) = m$. Then, $\ell(G) = \lceil \frac{\mathsf{D}(G)}{m} \rceil$.*

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Appendix A. Proof of Lemma 2.2

For every sequence S over a finite abelian group G, let

$$\mathrm{h}(S) = \max\{\mathrm{v}_g(S), g \in G\}.$$

Lemma A.1. *(See [19, Proposition 4.2.6].) If S is a sequence over G of length $|S| \geqslant |G|$ then S contains a zero-sum subsequence T of length $|T| \in [1, \mathrm{h}(S)]$.*

Let $G = C_n$. For a sequence $S = (x_1 g) \cdot (x_2 g) \cdot \cdots \cdot (x_\ell g)$, where $g \in G \setminus \{0\}$ and $x_i \in [1, \mathrm{ord}(g)]$, let

$$L_g(S) = \sum_{i=1}^{\ell} x_i \in \mathbb{N}.$$

Lemma A.2. *(See [28,31].) Let $G = C_n$. Let S be a zero-sum free sequence of length greater that $\frac{n}{2}$. Then there exists $g \in G$ such that $L_g(S) < n$.*

Lemma A.3. *(See [29, Proposition 3].) Let $G = C_n$, and let S be a zero-sum free sequence over G. Suppose that there exists $g \in G$ such that $L_g(S) < \min\{2|S|, n\}$. Then:*

(a) $\mathrm{v}_g(S) \geqslant 2|S| - L_g(S)$.
(b) *For each integer $x \in [2|S| - L_g(S), L_g(S)]$, there exists a subsequence T of S with length at least $2|S| - L_g(S)$ such that $\sigma(T) = xg$.*

Proof Lemma 2.2. Without of loss generality assume that $\mathrm{v}_0(S) = \mathrm{h}(S)$. Let

$$S = 0^{\mathrm{h}(S)} T_1 T_2,$$

where T_1 is a zero-sum subsequence of S with nonzero terms and of maximum length, T_2 is zero-sum free.

Claim 1. $\mathrm{v}_0(S) + |T_1| = \mathrm{h}(S) + |T_1| \leqslant kn - 1$.

Proof. Assume to the contrary that $\mathrm{v}_0(S) + |T_1| \geqslant kn$. If $|T_1| < kn$, then $0^{kn-|T_1|} T_1$ is a zero-sum sequence of length kn, yielding a contradiction. Next assume that $|T_1| \geqslant kn$, by Lemma A.1 we can find a zero-sum subsequence T_1' of T_1, such that $kn \geqslant |T_1'| \geqslant kn - \mathrm{v}_0(S)$, and therefore $0^{kn-|T_1'|} T_1'$ is a zero-sum sequence of length kn, yielding a contradiction. This proves Claim 1. $\square$

By Claim 1 we have

$$|T_2| \geqslant n - t + 1 > \frac{n}{2}.$$

It follows from Lemma A.2 that there exists $g \in G$, such that $\frac{n}{2} < L_g(T_2) < n$. Let $T_1 = g^w(b_1 g) \cdot \cdots \cdot (b_q g)$, where $2 \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_q \leqslant n-1$ and $q \in \mathbb{N}_0$.

Claim 2. *Suppose that $b_{j_1}, \ldots, b_{j_m}$ are m terms such that the integer X satisfies $X \equiv b_{j_1} + \cdots + b_{j_m} \pmod n$ and $1 < X \leqslant L_g(T_2)$. Then $m \geqslant 2|T_2| - L_g(T_2)$ if $2|T_2| - L_g(T_2) \leqslant X \leqslant L_g(T_2)$ and $m \geqslant X$ if $1 < X < 2|T_2| - L_g(T_2)$.*

Proof. Let $T_1' = (b_{j_1} g) \cdot \cdots \cdot (b_{j_m} g)$. Then $\sigma(T_1') = L_g(T_1')g = Xg$. Let $2|T_2| - L_g(T_2) \leqslant X \leqslant L_g(T_2)$. By Lemma A.3, there is a subsequence T_2' of T_2 with length at least $2|T_2| - L_g(T_2)$ such that $X = L_g(T_2') \equiv \sum_{i=1}^m b_{j_i} \pmod n$, hence $\sigma(T_2') = \sum_{i=1}^m b_{j_i} g = \sigma(T_1')$. By the maximum of the length of T_1, we have $m \geqslant |T_2'| \geqslant 2|T_2| - L_g(T_2)$. Similarly, if $1 < X < 2|T_2| - L_g(T_2)$ then Xg can be expressed as the sum of X terms equal to g of T_2. The same argument as above gives $m \geqslant X$. This proves Claim 2. □

By Claim 2 we infer that

$$b_j > L_g(T_2), \quad j = 1, \ldots, q.$$

Indeed, if $1 < b_j \leqslant L_g(T_2)$ for some j then $1 \geqslant 2|T_2| - L_g(T_2)$ or $1 \geqslant b_j$, both of which are not true. Therefore $n - b_j < n - L_g(T_2) < \frac{n}{2}$, $j = 1, \ldots, q$.

Claim 3. $L_g(T_2) + \sum_{j=1}^q (n - b_j) < n$.

Proof. We may assume that $q \geqslant 1$. Suppose that Claim 3 is false, then $L_g(T_2) + \sum_{j=1}^q (n - b_j) \geqslant n$. Let $m \in [1, q]$ be the least integer such that there exist $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant q$ with $\sum_{i=1}^m (n - b_{j_i}) + L_g(T_2) \geqslant n$. Let

$$X = n - \sum_{i=1}^m (n - b_{j_i})$$

under the assumption that $\sum_{i=1}^m (n - b_{j_i}) + L_g(T_2) \geqslant n$. Then $X \leqslant L_g(T_2)$. By the minimality of m we infer that

$$X + (n - b_{j_t}) > L_g(T_2) \quad \text{for every } t \in [1, m].$$

Then $X > L_g(T_2) - (n - b_{j_t}) > L_g(T_2) - (n - L_g(T_2)) = 2L_g(T_2) - n \geqslant 1$ and hence $1 < X \leqslant L_g(T_2)$.

First assume that $1 < X < 2|T_2| - L_g(T_2)$. Claim 2 gives $m \geqslant X$. Recalling that $X + (n - b_{j_t}) > L_g(T_2)$, we have $n - b_{j_t} \geqslant L_g(T_2) + 1 - X > 0$ for $t = 1, \ldots, m$, which implies that

$$\begin{aligned} n = X + \sum_{i=1}^m (n - b_{j_i}) &\geqslant X + m\bigl(L_g(T_2) + 1 - X\bigr) \\ &\geqslant X + X\bigl(L_g(T_2) + 1 - X\bigr) = X\bigl(L_g(T_2) + 2 - X\bigr). \end{aligned}$$

Consider the quadratic function $f(t) = t^2 - (L_g(T_2) + 2)t + n$. We obtained $f(X) \geqslant 0$ for some $X \in \{2, \dots, 2|T_2| - L_g(T_2) - 1\}$. But the maximum of $f(t)$ on $\{2, \dots, 2|T_2| - L_g(T_2) - 1\}$ is $f(2) = n - 2L_g(T_2)$, and $n - 2L_g(T_2) < 0$. This is a contradiction.

Next assume that $2|T_2| - L_g(T_2) \leqslant X \leqslant L_g(T_2)$. By Claim 2 we have $m \geqslant 2|T_2| - L_g(T_2) > 1$. Then

$$L_g(T_2) + 1 \leqslant X + (n - b_{j_m}) = n - \left(\sum_{i=1}^{m-1} (n - b_{j_i})\right) \leqslant n - (m-1)$$
$$\leqslant n - \big(2|T_2| - L_g(T_2) - 1\big) = \big(n - 2|T_2|\big) + L_g(T_2) + 1.$$

This implies $n \geqslant 2|T_2|$, which yields a contradiction. This proves Claim 3. □

Recall that $T_1 = g^w (b_1 g) \cdot \dots \cdot (b_q g)$, where $2 \leqslant b_1 \leqslant b_2 \leqslant \dots \leqslant b_q \leqslant n-1$ and $q \in \mathbb{N}_0$. Since T_1 is zero-sum we have $w \equiv \sum_{j=1}^{q} (n - b_j) \pmod{n}$. By Claim 3,

$$0 \leqslant \sum_{j=1}^{q} (n - b_j) < n \quad \text{and thus } q < n.$$

Let $w = rn + w'$, where $0 \leqslant w' \leqslant n - 1$. Then

$$w' = \sum_{j=1}^{q} (n - b_j) \geqslant q.$$

Hence $L_g(T_2) + w' = L_g(T_2) + \sum_{j=1}^{q} (n - b_j) < n$. Since $L_g(T_2) \geqslant \mathrm{v}_g(T_2) + 2(|T_2| - \mathrm{v}_g(T_2))$ and $w = \mathrm{v}_g(T_1) = \mathrm{v}_g(S) - \mathrm{v}_g(T_2)$, we have

$$n - 1 \geqslant L_g(T_2) + w' \geqslant \mathrm{v}_g(T_2) + 2\big(|T_2| - \mathrm{v}_g(T_2)\big) + w - rn = 2\big(|T_2| + w\big) - \mathrm{v}_g(S) - rn. \tag{A.1}$$

Also we have

$$kn - 1 \geqslant \mathrm{v}_0(S) + |T_1| = \mathrm{v}_0(S) + w + q \geqslant 2\big(\mathrm{v}_0(S) + q\big) - \mathrm{v}_0(S) + rn. \tag{A.2}$$

Adding (A.1) and (A.2) and noting that $\mathrm{v}_0(S) + q + w + |T_2| = |S| = (k+1)n - t$, we obtain that $\mathrm{v}_g(S) + \mathrm{v}_0(S) \geqslant (k+1)n - 2t + 2$. Take $a = 0$ and $b = g$ and we are done.

Next assume that $1 < t < \frac{n+5}{3}$. Assume that $\mathrm{v}_a(S) + \mathrm{v}_b(S) \geqslant (k+1)n - 2t + 2$ and $\mathrm{v}_c(S) + \mathrm{v}_d(S) \geqslant (k+1)n - 2t + 2$. By Claim 1 we infer that $\mathrm{v}_g(S) \leqslant kn - 1$ for every $g \in \{a, b, c, d\}$, and hence $\mathrm{v}_g(S) \geqslant n - 2t + 3$ for every $g \in \{a, b, c, d\}$. If $\{a, b\} \neq \{c, d\}$, without loss of generality assume that $c \notin \{a, b\}$, then $\mathrm{v}_a(S) + \mathrm{v}_b(S) + \mathrm{v}_c(S) \geqslant (k+1)n - 2t + 2 + (n - 2t + 3) > (k+1)n - t = |S|$, yielding a contradiction. Therefore $\{a, b\}$ is the unique pair holding (2.1). □

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